Comparison of an Analytical and Numerical Solution for the Landmine Detection Problem

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Abstract: Acoustic landmine detection is accomplished using a loud speaker as an airborne source to generate low-frequency waves that enter the soil at a certain incident angle. At a specific frequency, the landmine will "vibrate" at resonance, imparting a certain velocity on the soil particles above it that is detected by a scanning Laser Doppler Vibrometer system. The ability to mathematically predict the soil surface velocity plots created from these experiments would enable the technology to be implemented faster in more challenging environments. An analytical solution[1] was determined and has been compared to experimental results.[2] However, the analytical problem demands significant time and computational resources. A problem identical to the analytical solution was implemented with COMSOL[TM], and has significantly reduced the computational time and resources necessary to find a solution while remaining accurate to the analytical result.

Keywords: Acoustic, Analytical, Landmine

1. Introduction

The analytical mathematical model[1] of the buried landmine detection problem involved solving two Helmholtz equations in a 2-layer waveguide, subject to boundary conditions appropriate for mine detection (Fig. 1). In the atmospheric layer, a point source (delta function) represented the loudspeaker. The soil was modeled as a finite porous layer. The top plate of the buried landmine was represented as a circular elastic membrane stretched flush over a cylindrical cavity in a rigid substrate beneath a porous layer. Mathematically, the homogeneous Helmholtz equation was used in the porous layer, with a Green's function representation of the membrane response. In the analytical soil resonance predictions[1], pressure was plotted as a function of frequency, and the resonances appear as local maximums and minimums.

In comparing the experimental[2] and analytical[1] results, only the frequency at which the resonances occur is of importance, since this

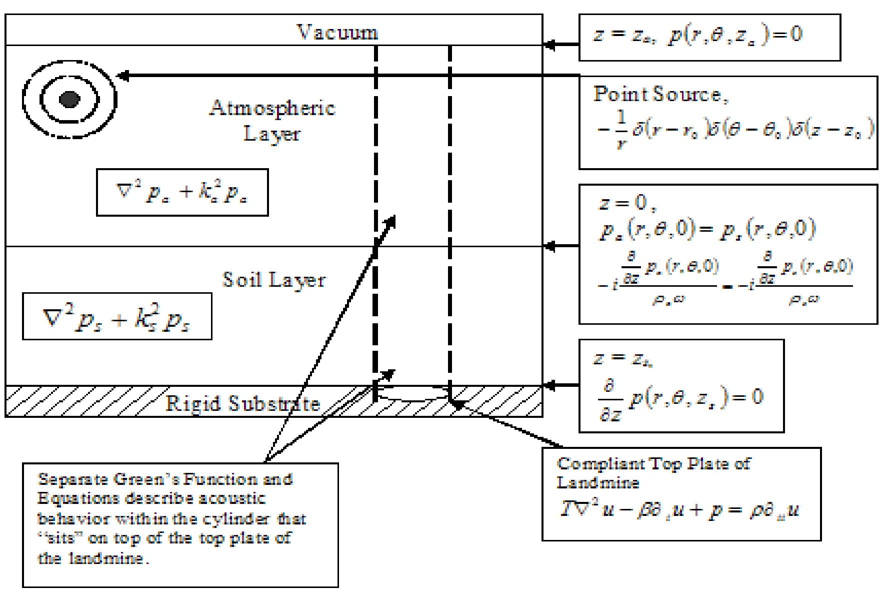

Figure 1. A schematic of the mathematical model developed to represent the landmine detection problem.

is what must be predicted mathematically to aid the experimenter in the field. Similarly, in comparing the COMSOL™ model developed in this paper to the analytical model, the resonant frequencies are of greater importance than the predicted amplitude of the response. Several COMSOL™ models have been run for comparison with the analytical models that were previously developed.[1]

2. Theory

Derivation of the three-dimensional wave equation in an ideal gas was performed for the analytical computation,[3,4] based on conservation of mass and conservation of momentum arguments. Assuming time-harmonic motion, the wave equation becomes

$$\nabla^2 P + k^2 P = 0,$$

a Helmholtz equation, with $k = \frac{\omega}{c}$, where ω is the angular frequency and c is the sound speed in the medium.

Morse and Ingard[3] argue that modifications are necessary for sound transmission in a porous media. Thus, the wavenumber (k), in the soil (subscripted 's') becomes

$$k_s = \sqrt{\frac{\omega^2 + \frac{i\omega\sigma}{\rho_p}}{c_s^2}},$$

where σ is the flow resistivity and ρ_p is the effective density in the soil.

Finally, the delta 'function' is customarily used to represent a point source radiating spherically. A plane wave representation was also considered but was ultimately rejected due to the close proximity of the source, target, and area of interest.

3. Governing Equations and Methods

The Helmholtz equation described in the previous section was converted into cylindrical coordinates for the analytical analysis[1] and separation of variables was applied to the partial differential equations. Note that the area outside the cylinder in Fig. 1 retains the mathematics from a two layer waveguide.[1,5,6,7] The Green's function was derived for a membrane, representing the top plate of the landmine. Continuity conditions were imposed between the cylinder containing the mine and the waveguide to find the analytical solution to the problem.

3.1 Eigenvalues

A change of variables was performed to force the eigenvalues (ζ_{mn} and τ_{mn}) to occur at predictable intervals for ease of computation. In order to perform these computations, parameters from D. Velea, R. Waxler, and J. Sabatier were used.[2] The equation defining the eigenvalues is

$$\begin{aligned}
&-\frac{\rho_s\omega^2}{T}\frac{1}{\left(k^2-\zeta_{mn}\right)J_m(ka)} \\
&\times\left(\begin{aligned}&\frac{\rho_s\tau_{mn}}{z_a\rho_a\sqrt{k_s^2-\zeta_{mn}}}\cos(\tau_{mn})\sin\left(z_s\sqrt{k_s^2-\zeta_{mn}}\right)\\ &-\sin(\tau_{mn})\cos\left(z_s\sqrt{k_s^2-\zeta_{mn}}\right)\end{aligned}\right) \\
&\times\left(J_m\left(\sqrt{\zeta_{mn}}a\right)I_1 - J_m(ka)I_2\right) + I_2 \\
&\times\left(\begin{aligned}&\frac{\rho_s\tau_{mn}}{z_a\rho_a}\cos(\tau_{mn})\cos\left(z_s\sqrt{k_s^2-\zeta_{mn}}\right)\\ &+\sqrt{k_s^2-\zeta_{mn}}\sin(\tau_{mn})\sin\left(z_s\sqrt{k_s^2-\zeta_{mn}}\right)\end{aligned}\right) = 0
\end{aligned}$$

for $m = 0,1,2,\ldots$, where

$$\begin{aligned}
I_1 &= \int_0^a rJ_m(kr)J_m\left(\sqrt{\zeta_{mn}}r\right)dr \\
&= \frac{a}{\zeta_{mn}-k^2}\left[\begin{aligned}&\sqrt{\zeta_{mn}}J_m(ka)J_{m+1}\left(\sqrt{\zeta_{mn}}a\right)\\ &-kJ_{m+1}(ka)J_m\left(\sqrt{\zeta_{mn}}a\right)\end{aligned}\right] \\
I_2 &= \int_0^a rJ_m^2\left(\sqrt{\zeta_{mn}}r\right)dr = \frac{a^2}{2}\left[\begin{aligned}&J_m^2\left(\sqrt{\zeta_{mn}}a\right)\\ &+J_{m+1}^2\left(\sqrt{\zeta_{mn}}a\right)\end{aligned}\right] \\
&\quad -\frac{a}{\sqrt{\zeta_{mn}}}\left(mJ_{m+1}\left(\sqrt{\zeta_{mn}}a\right)J_m\left(\sqrt{\zeta_{mn}}a\right)\right).
\end{aligned}$$

(See appendix for variable definitions)

3.2 Analytical Solution

Let $D = \{(r,\theta): r \le a\}$ be the area occupied by a membrane with a point source at (r_0,θ_0,z_0), $a < r_0$ with an atmosphere satisfying a pressure release condition at $z = z_a$.

Assume that total pressure is the sum of incident and scattered pressure. The incident pressure in air-soil waveguide in the absence of a membrane is

$$p_i(r,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}-\frac{i}{2\varepsilon_m\Phi_n}H_m^{(1)}\left(\sqrt{\mu_n}r_0\right)\times J_m\left(\sqrt{\mu_n}r\right)\cos(m(\theta_0-\theta))\chi(z_0)\times Z_n(z_0)Z_n(z),\ r<r_0$$

$$p_i(r,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}-\frac{i}{2\varepsilon_m\Phi_n}H_m^{(1)}\left(\sqrt{\mu_n}r\right)\times J_m\left(\sqrt{\mu_n}r_0\right)\cos(m(\theta_0-\theta))\chi(z_0)\times Z_n(z_0)Z_n(z),\ r>r_0\quad ,$$

where

$$\varepsilon_m=\begin{cases}2 & m=0\\ 1 & m>0\end{cases}.$$

Using the eigenvalues and radial continuity conditions, the final velocity was found to be

$$w=\frac{1}{i\omega\rho_a}\partial_z p=\frac{1}{i\omega\rho_a}\times\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\sqrt{\zeta_{mn}}r\right)A_{mn}\cos(m\theta)\phi'_{mn}(z)$$

for $r<a$, where $\phi_{mn}(z)=\sin(\Omega_a(z-z_a))$ and $\phi'_{mn}(z)=\Omega_a\cos(\Omega_a(z-z_a))$.

The coefficients, A_{mn}, are computed from an infinite set of algebraic equations that must be solved numerically from

$$\sum_{l=1}^{\infty}\left[\begin{array}{l}\sqrt{\zeta_{lm}}J_{m+1}\left(\sqrt{\zeta_{lm}}a\right)H_m^{(1)}\left(\sqrt{\mu_n}a\right)\\ -\sqrt{\mu_n}J_m\left(\sqrt{\zeta_{lm}}a\right)H_{m+1}^{(1)}\left(\sqrt{\mu_n}a\right)\end{array}\right]I_{mnl}A_{mn}=-\frac{i\sqrt{\mu_n}}{2\varepsilon_m}\chi(z_0)Z_n(z_0)H_m^{(1)}\left(\sqrt{\mu_n}r_0\right)\times\left[\begin{array}{l}J_{m+1}\left(\sqrt{\mu_n}a\right)H_m^{(1)}\left(\sqrt{\mu_n}a\right)\\ -J_m\left(\sqrt{\mu_n}a\right)H_{m+1}^{(1)}\left(\sqrt{\mu_n}a\right)\end{array}\right]\quad ,$$

where

$$I_{mnl}=\int_{z_s}^{z_a}\chi(z)\phi_{mn}(z)Z_n(z)dz=\int_{z_s}^{0}\rho_a\phi_{mn,s}(z)Z_{n,s}(z)dz+\int_{0}^{z_a}\rho_s\phi_{mn,a}(z)Z_{n,a}(z)dz\quad ,$$

$$\chi(z)=\begin{cases}\rho_s & 0<z\le z_a\\ \rho_a & z_s\le z<0\end{cases},$$

and

$$Z_{n,a}(z)=B\cos\left(\sqrt{k_s^2-\mu_n}z_s\right)\times\sin\left(\sqrt{k_a^2-\mu_n}(z_a-z)\right),\ 0\le z\le z_a$$

$$Z_{n,s}(z)=B\sin\left(\sqrt{k_a^2-\mu_n}z_a\right)\times\cos\left(\sqrt{k_s^2-\mu_n}(z-z_s)\right),\ z_s\le z<0\quad .$$

4. Numerical Model

The numerical model was created in COMSOL™ using the pressure subsection of the acoustics module. A twenty meter long and ten meter high rectangle was drawn to represent the atmosphere, and a twenty meter long and 0.075 meter deep rectangle was drawn to represent the soil (Fig. 2). A point source was implanted in the atmospheric layer, at (0.2, 2), corresponding to the analytical problem. A "mine" was placed beneath the soil layer, drawn as a 0.03 by 0.07 rectangle placed at (0, 0.105).

The subdomain settings of density, ρ, and sound speed through the medium, c, were consistent with the analytical solution, with the exception of the membrane sound speed. This was approximated from other plastics.[8] The expression for the wavenumber in the membrane is analytically given by

$$k=\sqrt{\frac{\rho_m\omega^2+i\beta_m\omega}{T}},$$

where $\beta_m=\beta$ from Fig. 1. The boundary settings were all set to continuity for the internal boundaries, and "Sound Hard Boundary" for the exterior boundaries. This was consistent with the analytical problem. The point at (0.2, 2) was set as a "Power Source" using the point settings, with a strength of 1 W.

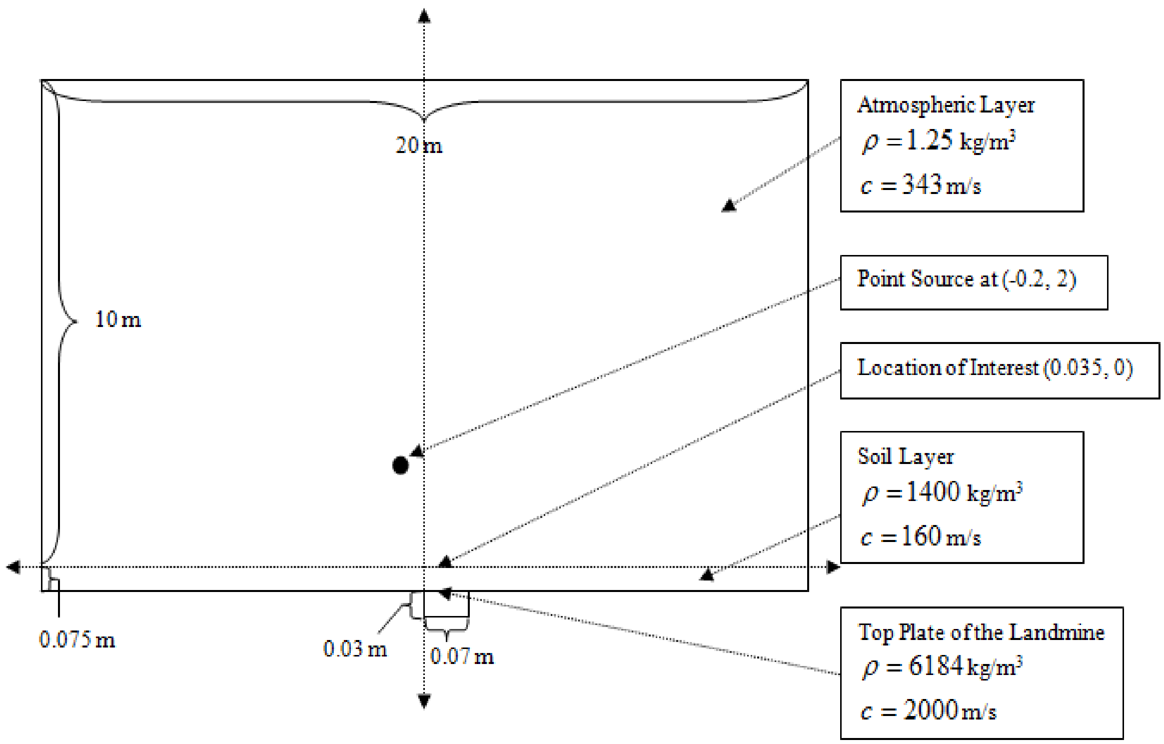

Figure 2. The COMSOL representation of the analytical problem represented in Fig. 1. The base model is exactly as shown in the figure. The extended model increases the depth of the soil layer from 0.075 m to 10 m, extending the soil layer while the mine remains in the same position.

5. Experimental Results

Preliminary results focusing on small frequency intervals around analytically predicted resonances showed expected behavior. However, once the frequency domain was expanded to the full range of 80 Hz to 300 Hz, these apparent "resonances" disappeared, indicating that they were small fluctuations in the pressure.

The amplitude of the response is significantly affected by the power of the source. Although the delta function was used analytically, a point source with unit power was applied in the numerical model. This causes variation in the amplitude of the response, but does not affect the resonant frequencies.

To improve the solution for comparison, certain changes were implemented. Damping was added in the soil and mine subdomains to increase the frequency span of the resonances. Since the analytical model incorporated damping in these two layers using complex wavenumbers, this implementation was valid, although little damping was used to produce the analytical result. Values and the types of damping were also varied in attempt to reproduce the analytical solution and ascertain the effects of the specific values on the problem.

Changes were also made in the processing and postprocessing of the solution for better comparison. Rather than plotting the pressure, the velocity was plotted to allow direct comparison with the analytical solution. While COMSOL™ Script could have been used to plot the parameterized frequency values, the point selection in the cross sectional plot parameters menu was utilized to create the plots quickly and accurately with respect to the point of interest.

6. Discussion

Comparison of the analytical solution (Fig. 3) to the base model (Fig. 4) showed good agreement on the soil-membrane resonance that occurred at $f \approx 280$ Hz. Velea, Sabatier, and

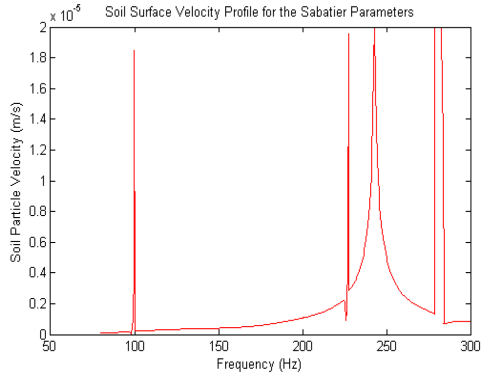

Figure 3. Analytical solution of the landmine detection problem for the Sabatier[2] parameters.

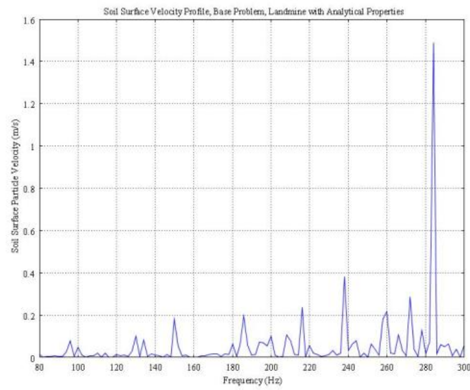

Figure 4. COMSOL solution for the landmine detection problem for the Sabatier[2] parameters.

Waxler[2] note that this is the first acoustical resonance, defined as the frequency at which "the layer of soil between the top of the mine and the surface of the soil vibrates at maximum amplitude."[2] The COMSOL™ model also remained stable at this point, whereas the analytical model was difficult to keep stable around this point.

The mechanical resonances (defined as "the resonant frequencies of any mechanical system for which the input mechanical reactance goes to zero"[2]) did not appear in the COMSOL™ model. Since these resonances were quite narrow in the analytical model ($f \approx 100$ Hz and $f \approx 225$ Hz), it is possible that these resonances were too narrow to detect, even with large damping and small sampling (0.25 Hz).

The wide resonance at $f \approx 240$ Hz was determined to be a soil resonance,[1] which was

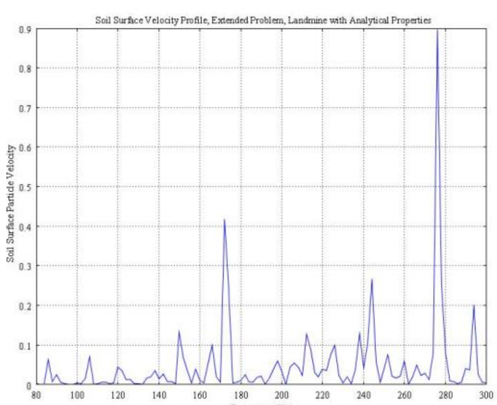

Figure 5. COMSOL solution for the extended landmine detection problem.

captured in the COMSOL™ model as well. The extended model with the same analytic properties (Fig. 5) also demonstrated strong resonances at $f \approx 240$ Hz and $f \approx 280$ Hz.

However, another strong resonance is observed at $f \approx 175$ Hz, which is not seen in the base model. This apparent resonance was consistent without damping, or with other types of damping. Mathematically, the extended problem is very difficult to solve and was not approached analytically. It is possible that this resonance is from the column of soil beneath the mine, or an interaction between the sides of the mine and the soil. However, the presence of the two expected resonances reinforces a certain confidence in the solution.

In Fig. 6, the predicted effects of the landmine size are considered. In the analytical solution, the resonant frequency decreases and the response amplitude increases with an increase in landmine size. In the profile (a significant increase in the mine size), the large resonance appears to shift to $f \approx 220$ Hz. This shift of approximately 60 Hz is not proportional to the analytical shift; however, the analytical data does not extend to $a = 0.40$ m, and shows indications of non-linear behavior.

The predicted effects of the mine depth are shown in Fig. 7. Analytical predictions indicate that increasing soil depth increases the resonant frequency. There were no conclusive predictions about the amplitude from the soil depth. A frequency shift has appeared to occur, but a second resonance has also appeared that cannot be attributed to the soil. Analytically, soil resonances are predicted at $f \approx 140, 230, 320$ Hz,

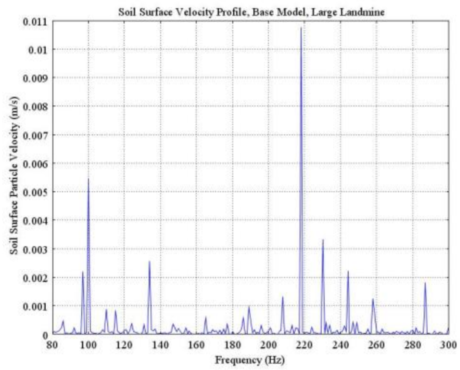

Figure 6. Soil surface profile after increasing the landmine diameter to 0.04 meters.

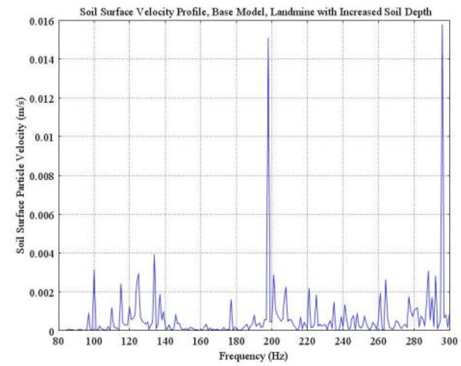

Figure 7. Soil surface profile after increasing the soil layer depth to 1 meter.

which do not account for the extra resonances at that depth. It is possible that the additional depth created another acoustical resonance.

7. Conclusions

In the field of physical mine detection, information speed and clarity is very important. The analytical solution of the landmine detection problem in Fig. 1 took approximately 6-7 months to derive, understand, code, and analyze solutions. COMSOL™ reduced this time to approximately an hour per solution, once the model had been developed.

The solutions produced by COMSOL™ were comparable to the analytical solution. Strong acoustical landmine resonances were predicted by the analytical[1] and COMSOL™ models, which matched the experimental data.[2] Although there were no mechanical resonances predicted by the COMSOL™ models, the matching acoustic and soil resonances are quite significant across the experimental, analytical, and numerical implementations.

The solutions for comparison of the landmine size and depth produced some interesting results that will require more analysis of the analytical solution. Due to the computational time, the analytical comparison models were only run for small frequency ranges. The more detailed COMSOL™ models have indicated more complicated behaviors than the analytical models that need to be investigated further.

Overall, the COMSOL™ models developed in this analysis and comparison have confirmed the analytical solution, while raising interesting questions to further research in this area. The computational processing time of the analytical model restricted the frequencies of consideration in the model analysis, which prevented a full frequency analysis with regard to the mine properties. The COMSOL™ models have raised these important questions, which will help further research in the field of acoustic landmine detection.

8. References

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10. Appendix

Table 1: Definition of Variables (In order of appearance)

Variable	Definition
P	Pressure (Pa)
k	Wavenumber (1/m), may be subscripted with 'a', atmospheric, or 's' soil (unsubscripted - membrane)
ω	Angular Frequency (Hz)
c	Sound Speed (m/s), may be subscripted with 'a', atmospheric, or 's' soil
σ	Flow Resistivity
ρ_p	Effective Density
$\zeta_{mn}, \tau_{mn}, \mu_n$	Eigenvalues
ρ	Density (kg/m^3), may be subscripted with 'a', atmospheric, 'm' membrane, or 's' soil
T	Membrane Tension
a	Diameter of the Landmine
z_a	Height of the atmospheric layer (m)
z_s	Depth of soil layer (m)
(r, θ, z)	Position of interest (m)
(r_0, θ_0, z_0)	Source location (m)
B	Arbitrary coefficient
β_m	Coefficient from the time derivative in the differential equation for the motion of a membrane exposed to pressure

Table 2: Properties of COMSOL Graphics

Figure Number	Notes
4	Damping in the soil - $R_f = 1000$ Damping in the mine - $\alpha = 100$
5	Damping in the soil - $R_f = 1000$ Damping in the mine - $\alpha = 100$
6	$a = 0.40$ m, damping is consistent with Fig. 4 and 5
7	$z_s = -1$ m, damping is consistent with Fig. 4 and 5