Application of Kelvin's Inversion Theorem to the Solution of Laplace's Equation over a Domain that Includes the Unbounded Exterior of a Sphere

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Abstract: One may cut the unbounded domain exterior to a solid body into two parts that share a common boundary in the form of a sphere of radius a, which I will call the *Reflecting Sphere*. The *Near Exterior* is exterior to the solid body but interior to the Reflecting Sphere while the *Far Exterior* is exterior to both. Suppose the velocity field in the Far Exterior has a velocity potential, ϕ. In the 1840s Kelvin showed that the change of position variable $\mathbf{r} \to \mathbf{q}$ defined by $\mathbf{r}/r = \mathbf{q}/q$ with $rq = a^2$ (in which $r = |\mathbf{r}|$ and $q = |\mathbf{q}|$) maps the Far Exterior to an *Inverted Far Exterior* (the interior of the Reflecting Sphere). Furthermore, if $\mathbf{r} \mapsto \phi$ is a solution of Laplace's equation in the $\mathbf{r}$-coordinates then $\mathbf{q} \mapsto \Phi$, in which $\phi = (q/a)\Phi$, is a solution of Laplace's equation in the $\mathbf{q}$-coordinates. Boundedness of both the Near Exterior and Inverted Far Exterior enables meshing in both geometries and simultaneous solution of the relevant partial differential equations provided one applies suitable compatibility conditions on the Reflecting Sphere. This paper develops techniques for accomplishing this task, presents a computational example, and compares ther results with an analytical solution.

Keywords: Unbounded domains, Harmonic functions, Kelvin Inversion, Numerical dipole

1. Introduction

1.1 On Kelvin Inversion and the Kelvin Inversion Theorem. Let a denote a given length scale. If $\mathbf{r} = \overrightarrow{OP}$ is the vector position of a point P relative to an origin O and $r := |\mathbf{r}|$ then the inequality $r > a$ describes the unbounded exterior of a sphere of radius a. Consider the change of variable $\mathbf{r} \mapsto \mathbf{q}$ defined by $\mathbf{r}/r = \mathbf{q}/q$, in which $\mathbf{q} = \overrightarrow{OQ}$ is the vector position of a point Q relative to the origin O, $q := |\mathbf{q}|$, and $rq = a^2$. Then a is the geometric mean of the lengths r and q. In particular, if $r > a$ we have $q < a < r$. The change of position variable $\mathbf{r} \mapsto \mathbf{q}$ therefore takes a point P exterior to a sphere of radius a (in a space where $\mathbf{r}$ denotes position) to an image point Q interior to that sphere (in a space where $\mathbf{q}$ denotes position). Kelvin noted the foregoing in the 1840s in two seminal publications

(References 1 and 2). Following tradition, I will refer to this association between $\mathbf{r}$ and $\mathbf{q}$ as *Kelvin Inversion*. I will also refer to the sphere of radius a in the present context as the *Reflecting Sphere*.

Let $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ be a right-handed set of constant unit vectors. Then the vectors $\mathbf{r}$ and $\mathbf{q}$ have expansions into components relative to $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$, *i.e.*

$$\mathbf{r} = \sum_{i=1}^{3} x_i \hat{\mathbf{i}}_i \quad , \quad \mathbf{q} = \sum_{i=1}^{3} q_i \hat{\mathbf{i}}_i \ . \qquad (1.1)_{1,2}$$

Now the ith scalar component of the vector equation $\mathbf{r}/r = \mathbf{q}/q$ is the scalar equation $x_i/r = q_i/q$, so $x_i = (r/q)q_i$. But $rq = a^2$, so

$$x_i = (a/q)^2 q_i \ , i \in \{1, 2, 3\} \ , \qquad (1.2)$$

Equation $(1.1)_1$ is therefore equivalent to

$$\mathbf{r} = \left(\frac{a}{q}\right)^2 \sum_{i=1}^{3} q_i \hat{\mathbf{i}}_i \ . \qquad (1.3)$$

In this way Kelvin Inversion defines a transformation $(q_1, q_2, q_3) \mapsto \mathbf{r}$.

Whenever there is a function $(q_1, q_2, q_3) \mapsto \mathbf{r}$ that expresses position, $\mathbf{r}$, in terms of a list of scalars, (q_1, q_2, q_3) those scalars constitute curvilinear coordinates in a curvilinear coordinate system. A curvilinear coordinate system is orthogonal if

$$\frac{\partial \mathbf{r}}{\partial q_i} \boldsymbol{\cdot} \frac{\partial \mathbf{r}}{\partial q_j} = 0 \quad \text{for all} \quad i \neq j \ . \qquad (1.4)$$

Given any orthogonal curvilinear coordinate system in three dimensions one associates each of the three coordinates with one of three scale factors, (h_1, h_2, h_3) defined by

$$h_1^2 = \frac{\partial \mathbf{r}}{\partial q_1} \boldsymbol{\cdot} \frac{\partial \mathbf{r}}{\partial q_1} \ , \ h_2^2 = \frac{\partial \mathbf{r}}{\partial q_2} \boldsymbol{\cdot} \frac{\partial \mathbf{r}}{\partial q_2} \ , \ h_3^2 = \frac{\partial \mathbf{r}}{\partial q_3} \boldsymbol{\cdot} \frac{\partial \mathbf{r}}{\partial q_3} \ . \qquad (1.5)_{1,2,3}$$

Let Δ_x denote the Laplace operator with respect to the cartesian coordinates (x_1, x_2, x_3), *i.e.*

$$\Delta_x := \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j \partial x_j} \ . \qquad (1.6)$$

Let $(x_1, x_2, x_3) \mapsto \phi$ be a twice differentiable scalar function. A classical result from the theory of orthogonal curvilinear coordinates asserts that

$$\Delta_x \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial q_2} \right) \right.$$
$$\left. + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right] . \quad (1.7)$$

In the particular case when KELVIN Inversion defines the curvilinar coordinates [as in (1.3)] one finds that the system is, indeed, orthogonal. One also finds that

$$h_1 = h_2 = h_3 := h = (a/q)^2 ,$$

so (1.7) simplifies to

$$\Delta_x \phi = \frac{1}{h^3} \sum_{j=1}^{3} \frac{\partial}{\partial q_j} \left(h \frac{\partial \phi}{\partial q_j} \right) ,$$

or

$$\Delta_x \phi = \left(\frac{q}{a} \right)^6 \sum_{j=1}^{3} \frac{\partial}{\partial q_j} \left[\left(\frac{a}{q} \right)^2 \frac{\partial \phi}{\partial q_j} \right] . \quad (1.8)$$

Note that

$$\frac{a}{q} \frac{\partial \phi}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\frac{a}{q} \phi \right) - \frac{\partial}{\partial q_j} \left(\frac{a}{q} \right) \phi$$

so

$$\left(\frac{a}{q} \right)^2 \frac{\partial \phi}{\partial q_j} = \frac{a}{q} \frac{\partial}{\partial q_j} \left(\frac{a}{q} \phi \right) - \frac{\partial}{\partial q_j} \left(\frac{a}{q} \right) \frac{a}{q} \phi .$$

Therefore

$$\frac{\partial}{\partial q_j} \left[\left(\frac{a}{q} \right)^2 \frac{\partial \phi}{\partial q_j} \right]$$
$$= \frac{\partial}{\partial q_j} \left[\frac{a}{q} \frac{\partial}{\partial q_j} \left(\frac{a}{q} \phi \right) \right] - \frac{\partial}{\partial q_j} \left[\frac{\partial}{\partial q_j} \left(\frac{a}{q} \right) \frac{a}{q} \phi \right]$$
$$= \frac{\partial}{\partial q_j} \left(\frac{a}{q} \right) \frac{\partial}{\partial q_j} \left(\frac{a}{q} \phi \right) + \frac{a}{q} \frac{\partial^2}{\partial q_j \partial q_j} \left(\frac{a}{q} \phi \right)$$
$$- \left[\frac{\partial^2}{\partial q_j \partial q_j} \left(\frac{a}{q} \right) \frac{a}{q} \phi + \frac{\partial}{\partial q_j} \left(\frac{a}{q} \right) \frac{\partial}{\partial q_j} \left(\frac{a}{q} \phi \right) \right] ,$$

which simplifies to

$$\frac{\partial}{\partial q_j} \left[\left(\frac{a}{q} \right)^2 \frac{\partial \phi}{\partial q_j} \right] = \frac{a}{q} \frac{\partial^2}{\partial q_j \partial q_j} \left(\frac{a}{q} \phi \right) - \frac{\partial^2 (a/q)}{\partial q_j \partial q_j} \frac{a}{q} \phi .$$
$$(1.9)$$

When one substitutes (1.9) into (1.8) an additional simplification is possible owing to the identity

$$\sum_{j=1}^{3} \frac{\partial^2 (1/q)}{\partial q_j \partial q_j} = 0 \quad , \quad (q \neq 0) . \quad (1.10)$$

If one substitutes (1.9) into (1.8) and simplifies by means of (1.10) one arrives at the result

$$\Delta_x \phi = \left(\frac{q}{a} \right)^6 \sum_{j=1}^{3} \frac{a}{q} \frac{\partial^2}{\partial q_j \partial q_j} \left(\frac{a}{q} \phi \right) . \quad (1.11)$$

Let Δ_q denote the LAPLACE operator with respect to the coordinates (q_1, q_2, q_3), i.e.

$$\Delta_q := \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j \partial q_j} . \quad (1.12)$$

Then (1.11) takes the more compact form

$$\Delta_x \phi = (q/a)^5 \Delta_q [(a/q) \phi] . \quad (1.13)$$

If $\Delta_x \phi = 0$, i.e. ϕ is a harmonic function of the coordinates (x_1, x_2, x_3) and if $q \neq 0$ [as already assumed in the derivation of (1.13)] then (1.13) reduces to $\Delta_q[(a/q)\phi] = 0$, i.e. $(a/q)\phi$ is a harmonic function of the coordinates (q_1, q_2, q_3). This last result is *Kelvin's Inversion Theorem.*

1.2 Objectives.

O1. Formulate and test a numerical method that fulfills two requirements, namely

   i. The method solves for a harmonic function over an unbounded domain; and

   ii. The only computational domains for which the method requires meshing are bounded.

O2. Compare the computational results with an analytic solution.

2. A classical problem

2.1 Analytical solution for the disturbance of a liquid from rest due to sudden acceleration of a submerged sphere. If one rewrites (1.10) by means of the abbreviation (1.12) it takes the compact form $\Delta_q(1/q) = 0$ provided $q \neq 0$. Similarly, we have the identity $\Delta_x(1/r) = 0$ provided $r \neq 0$. Readers familar with potential theory will know the function $(x_1, x_2, x_3) \mapsto 1/r$ by the name *simple source.* If one differentiates the identity $\Delta_x(1/r) = 0$ with respect to one of the cartesian coordinates, say x_3, one gets $\partial[\Delta_x(1/r)]/\partial x_3 = 0$, or equivalently,

$$\Delta_x[\partial(1/r)/\partial x_3] = 0 , \qquad (2.1)$$

which shows that the function $\partial(1/r)/\partial x_3$ is also a harmonic function of the coordinates (x_1, x_2, x_3). Readers familar with potential theory will know the function $(x_1, x_2, x_3) \mapsto \partial(1/r)/\partial x_3$ by the name *simple dipole.*

From the definition $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ one may deduce the identity

$$\partial r/\partial x_3 = x_3/r . \qquad (2.2)$$

In view of (2.2) we also have

$$\partial(1/r)/\partial x_3 = -r^{-2}(x_3/r) . \qquad (2.3)$$

Now let (r, ϑ, φ) be the spherical coordinates associated with the cartesian coordinates (x_1, x_2, x_3) in the usual way, *i.e.*

$$x_1 = r\sin\vartheta\cos\varphi , \; x_2 = r\sin\vartheta\sin\varphi , \; x_3 = r\cos\vartheta . \tag{2.4}_{1,2,3}$$

Then (2.3) takes the form

$$\partial(1/r)/\partial x_3 = -r^{-2}\cos\vartheta . \qquad (2.5)$$

If a fluid is incompressible and in irrotational motion in a simple connected domain then its velocity field $\mathbf{r} \mapsto \mathbf{v}$ satisfies

$$\mathbf{v} = \nabla_x\phi \qquad (2.6)$$

for a harmonic function, ϕ, *i.e.* a *velocity potential.* Here,

$$\nabla_x = \sum_{i=1}^3 \hat{\mathbf{i}}_i \frac{\partial}{\partial x_i} . \qquad (2.7)$$

A constant multiple of the simple dipole is an example of a velocity potential, say ϕ_d. Thus

$$\phi_d = -Cr^{-2}\cos\vartheta \qquad (2.8)$$

in which C is a constant to be determined by boundary conditions. If a solid sphere of radius a_s is accelerated suddenly from rest a suitable boundary condition would require that the wall be impermeable. If, say, the velocity of the sphere equals $Q\hat{\mathbf{i}}_3$ after the sudden acceleration, in which $Q > 0$ is a given sphere speed, then the impermeability condition asserts that

$$[(\mathbf{r}/r) \cdot (\nabla_x\phi_d - Q\hat{\mathbf{i}}_3)]_{r=a_s} = 0 , \qquad (2.9)$$

or

$$[\partial\phi_d/\partial r - Qx_3/r]_{r=a_s} = 0 ,$$

or

$$[\partial\phi_d/\partial r - Q\cos\vartheta]_{r=a_s} = 0 ,$$

or

$$[\partial\phi_d/\partial r]_{r=a_s} = Q\cos\vartheta . \qquad (2.10)$$

If one substitutes (2.8) into (2.10) the result reduces to an equation for C, namely

$$C = Qa_s^3/2 . \qquad (2.11)$$

Equation (2.8) then becomes

$$\phi_d = -\frac{Qa_s^3}{2r^2}\cos\vartheta = -\frac{1}{2}\left(\frac{a_s}{r}\right)^3 Qx_3 . \qquad (2.12)$$

From (2.12) one may verify that $|\mathbf{v}| = |\nabla_x\phi_d| \to 0$ as $r \to \infty$.

One expedite comparison with numerical results by expressing the components of the fluid velocity vector, $\mathbf{v} = \nabla_x\phi_d$, in cylindrical coordinates. To this end let (ϖ, φ, z) be cylinderical coordinates associated with the cartesian coordinates (x_1, x_2, x_3) in the usual way, *i.e.*

$$x_1 = \varpi\cos\varphi , \; x_2 = \varpi\sin\varphi , \; x_3 = z . \tag{2.13}_{1,2,3}$$

It follows that

$$x_1^2 + x_2^2 = \varpi^2 , \qquad (2.14)$$

so $x_1^2 + x_2^2 + x_3^2 = \varpi^2 + z^2$, or

$$r^2 = \varpi^2 + z^2 . \qquad (2.15)$$

Equation (2.12) for the velocity potential ϕ_d becomes

$$\phi_d = -\frac{Qa_s^3}{2}\frac{z}{(\varpi^2 + z^2)^{3/2}} . \qquad (2.16)$$

The derivatives $\partial\phi_d/\partial\varpi$ and $\partial\phi_d/\partial z$ represent the lateral and axial components of the velocity vector, respectively.

2.2 Partition of the domain into near and far exteriors. Let the radius, a_s, of a solid sphere be given. Let the radius, a, of the Reflecting Sphere in KELVIN Inversion satisfy $a > a_s$. I will employ the terms *Near Exterior* and *Far Exterior* to denote the regions where $a_s < r < a$ and $r > a$, respectively. Employing the foregoing notation for KELVIN Inversion I will employ the term *Inverted Far Exterior* to denote the region $q < a$.

2.3 Analytic solution in the Inverted Far Exterior. Let the expression $(a/q)\phi$ as in equation (1.13) have the abbreviation

$$\Phi := (a/q)\phi . \qquad (2.17)$$

Then KELVIN's Inversion theorem, as stated in the text after (1.13), asserts that $(q_1, q_2, q_3) \mapsto \Phi$ is a solution of $\nabla_q \Phi = 0$ in the Inverted Far Exterior, $q < a$, whenever $(x_1.x_2, x_3) \mapsto \phi$ is a solution of $\nabla_x \phi = 0$ in the (ordinary) Far Exterior, $r > a$. Consider the outermost equality in (2.12) in the form

$$\phi_d = -\frac{Qa_s^3}{2}\left(\frac{1}{r}\right)^2 \frac{x_3}{r} . \qquad (2.18)$$

But $x_3/r = q_3/q$ [see the text between (1.1) and (1.2)]. In the mean time $rq = a^2$ implies that $r^{-2} = q^2/a^4$. Equation (2.18) therefore becomes

$$\phi_d = -\frac{Qa_s^3}{2}\frac{q^2}{a^4}\frac{q_3}{q} ,$$

or

$$\phi_d = -\frac{Q}{2}\left(\frac{a_s}{a}\right)^3 \frac{q}{a} q_3 . \qquad (2.19)$$

In view of (2.17) the particular form that the harmonic function Φ takes in the Inverted Far Exterior reduces to the simple linear function

$$\Phi_d = -\frac{Q}{2}\left(\frac{a_s}{a}\right)^3 q_3 . \qquad (2.20)$$

2.4 Boundary-value problems for the near and far exteriors

2.4.1 Partial differential equations for the near and far exteriors. As the summary in the text after

(2.17) asserts the differential equation satisfied by $\mathbf{r} \mapsto \phi$ in the Near Exterior ($r < a$) is

$$\nabla_x \phi = 0 . \qquad (2.21)$$

Now the cylinderical coordinate transformation $(\varpi, \varphi, z) \mapsto (x_1, x_2, x_3)$ is a special case of the generic orthogonal curviliear coordinate transformation $(q_1, q_2, q_3) \mapsto (x_1, x_2, x_3)$. In the the axissymmetric case—in which nothing depends upon the azimuthal coordinate φ—the corresponding form of (1.7), simplifies to the classical result

$$\Delta_x \phi = \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \phi}{\partial \varpi}\right) + \frac{\partial^2 \phi}{\partial z^2} ,$$

or, upon multiplication by ϖ,

$$\varpi \Delta_x \phi = \frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \phi}{\partial \varpi}\right) + \frac{\partial}{\partial z}\left(\varpi \frac{\partial \phi}{\partial z}\right) .$$

The left member vanishes by (2.21), so

$$\frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \phi}{\partial \varpi}\right) + \frac{\partial}{\partial z}\left(\varpi \frac{\partial \phi}{\partial z}\right) = 0 . \qquad (2.22)$$

In a similar way the differential equation satisfied by $\mathbf{q} \mapsto \Phi$ in the Inverted Far Exterior ($q < a$) becomes

$$\frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \Phi}{\partial \varpi}\right) + \frac{\partial}{\partial z}\left(\varpi \frac{\partial \Phi}{\partial z}\right) = 0 . \qquad (2.23)$$

2.4.2 Impermeable-wall condition for the near exterior. Let $\hat{\mathbf{n}}$ denote the unit vector directed out of the Near Exterior on the inner boundary of that domain. Then $\hat{\mathbf{n}} = -[\mathbf{r}/r]_{r=a_s}$. If one multiplies equation (2.9) by -1 that equation becomes

$$\hat{\mathbf{n}} \cdot [(\nabla_x \phi_d)_{r=a_s} - Q\hat{\mathbf{i}}_3] = 0 ,$$

or

$$\partial \phi_d/\partial n - Q\hat{\mathbf{n}} \cdot \hat{\mathbf{i}}_3 = 0 . \qquad (2.24)$$

If one denotes by (n_1, n_2, n_3) the scalar components of $\hat{\mathbf{n}}$ relative to $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ then (2.24) becomes

$$\partial \phi_d/\partial n = Qn_3 = 0 . \qquad (2.25)$$

or, upon multiplication by ϖ,

$$\varpi \partial \phi_d/\partial n = \varpi Qn_3 . \qquad (2.26)$$

2.4.3 Continuity of velocity potential at the Reflecting Sphere. Recall from §**1.1** that one of the defining conditions of KELVIN inversion is $rq = a^2$. On the reflecting sphere $r = q = a$ and equation (2.17) reduces to

$$\Phi_{RS} = \phi_{RS} \; , \qquad (2.27)$$

in which the subscript $(\;)_{RS}$ denotes evaluation on the Reflecting Sphere.

2.4.4 Continuity of normal velocity at the Reflecting Sphere. One may arrange (2.17) in the form

$$\phi = (q/a)\Phi \qquad (2.28)$$

so

$$\frac{\partial \phi}{\partial q} = \frac{q}{a}\frac{\partial \Phi}{\partial q} + \frac{1}{a}\Phi \; . \qquad (2.29)$$

But $\partial/\partial q = (dr/dq)\partial/\partial r$ in which the identity $rq = a^2$ implies that $r = a^2/q$. Therefore $dr/dq = -a^2/q^2$ and (2.29) becomes

$$-\frac{a^2}{q^2}\frac{\partial \phi}{\partial r} = \frac{q}{a}\frac{\partial \Phi}{\partial q} + \frac{1}{a}\Phi \; ,$$

or, equivalently,

$$\frac{q}{a}\frac{\partial \Phi}{\partial q} = -\frac{a^2}{q^2}\frac{\partial \phi}{\partial r} - \frac{1}{a}\Phi \; . \qquad (2.30)$$

If one evaluates (2.30) on the Reflecting Sphere and recalls that $r = q = a$ on that boundary one obtains

$$\left(\frac{\partial \Phi}{\partial q}\right)_{RS} = -\left(\frac{\partial \phi}{\partial r}\right)_{RS} - \frac{1}{a}\Phi_{RS} \; , \qquad (2.31)$$

or, upon multiplication by ϖ,

$$\varpi\left(\frac{\partial \Phi}{\partial q}\right)_{RS} = -\varpi\left(\frac{\partial \phi}{\partial r}\right)_{RS} - \frac{\varpi}{a}\Phi_{RS} \; . \qquad (2.32)$$

3. COMSOL simulation for the velocity potential

One may replicate the results of the §**4** below by building a COMSOL model tree according to the following scheme.

1. After lauching COMSOL respond the the usual interrogatories as follows: For Space Dimenison, choose 2D axisymmetric; For Add Physics, choose Mathematics > Classical PDEs > Poisson's Equation (poeq)*.

2. Construct the first branch of the model tree to solve for the velocity potential, ϕ, in the Near Exterior. To this end change the name of the dependent variable from the default `u` to `phi` and specify units appropriate to the velocity potential (*i.e.* (`m^2/s`) for the dependent variable and `m/s` for the source term).

3. For Study Type, choose Stationary and click Finish. Select the node Poisson's Equation 1. Expand the heading Diffusion Coefficient, c, and enter `r` in the c-field. Expand the heading Source Term, f, and enter 0 in the f-field. This instruction will implement equation (2.22) above.

4. Add a second branch of the model tree with which to solve for Φ in the Inverted Far Exterior. To do so right-click the root node and choose Add Model. Respond to the interrogatories as in steps 1 & 2 above but this time name the dependent variable `Phi` and click Finish without adding a second Study.

5. Right-click Global Definitions and choose Parameters. Denote the first parameter by `a_s`. Assign it the value `1[m]` and the description Radius of solid sphere. Denote the second parameter by `radiusRatio`. Assign it the value `1.5`. Denote the third parameter by `a`. Assign it the value `radiusRatio*a_s` and the description Radius of reflecting sphere. Denote the fourth parameter by `Q`. Assign it the value `1[m/s]` and the description Speed of solid sphere (after sudden acceleration from rest).

6. Rename Model 1 Near Exterior and Model 2 Inverted Far Exterior.

7. For the Geometry in the Near Exterior build two circles of radii `a` and `a_s` and form their Boolean Difference. For the Geometry in the Inverted Far Exterior build just one circle of radius `a`.

8. To expedite later statements of boundary conditions introduce explicit selections under the local definitions lists for each model, namely

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* In 2D Axisymmetric models COMSOL's Physics Interface Laplace Equation (lpeq) appears to solve the equation $u_{rr} + u_{zz} = 0$ for u, of which equation (2.22) is *not* a special case.

selections for the Solid Sphere and the Reflecting Sphere under the local definitions list for the Near Exterior and for the Reflecting Sphere under the local definitions list for the Inverted Far Exterior.

9. To enable exhange of information between the two branches of the model tree introduce a Model Coupling Operator of General Extrusion type under the local definitions list for the Near Exterior. It should assign itself the default name `genext1`. Under Geometric entity level select Boundary and under Selection choose Reflecting Sphere. Accept the default formulas in the fields labeled Destination map. The effect is to tell COMSOL that the values on the Reflecting Sphere of $(x_1^2 + x_2^2)^{1/2}$ coincide with those of $(q_1^2 + q_2^2)^{1/2}$ and the values of x_3 coincide with those of q_3. In a similar way introduce a Model Coulpling Operator of General Extrusion type under the local definitions list for the Near Exterior. It should assign itself the default name `genext2`. All additional specifications track those of `genext1`.

10. Under the Near Exterior branch right-click the Poisson Equation Physics Interface and select Flux/Source. Choose Solid Sphere for the boundary and expand the heading Equation to display the definitions of the input parameters. In the field under Boundary Flux/Source enter `r*Q*nz`. This instruction will implement equation (2.26) above.

11. Under the Near Exterior branch right-click the Poisson Equation Physics Interface and select Dirichlet Boudary Condition. Choose Reflecting Sphere for the boundary. In the field under Dirichlet Boudary Condition > Prescribed value of `phi` enter `mod2.genext2(mod2.Phi)`. This instruction implements equation (2.27).

12. To enable reading of the boundary value of the radial velocity in the Near Exterior from within the Inverted Far Exterior, right-click the local definitions node in the Near Exterior branch and select Variables. Under Geometric entitiy level choose Boundary and under Selection choose Reflecting Sphere. Name a new variable `dphidR` and define its expression as `phir*(r/a)+phiz*(z/a)`.

13. Under the Inverted Far Exterior branch right-

click the Poisson Equation Physics Interface and select Flux/Source. Choose Reflecting Sphere for the boundary and expand the heading Equation to display the definitions of COMSOL's input parameters. In the field labeled Boundary Flux/Source enter

$$-\mathtt{r} * \mathtt{mod1.genext1}(\mathtt{mod1.dphidR})\,.$$

In the field labeled Boundary Absorbtion/Impedance Term enter `r/a`. These instructions will implement equation (2.32).

If one carries out steps 1–13 above and clicks Compute COMSOL will calculate the distributions of `phi` and `Phi` in the Near Exterior and the Inverted Far Exterior, respectively, and will furnish 2D and 3D Plot Groups for both. In order to expedite comparison between the computed results and the analytical ones I will include one more step.

14. Right-click the Definitions in the Near Exterior branch of the model tree and select Variables. For the Geometric entity level menu choose Domain and for the Selection menu choose all domains. In the first line of the Variables table enter `phiAnalytic` for the name. Enter

$$-(1/2) * \mathtt{Q} * \mathtt{a\_s}\hat{\ }3 * \mathtt{z} * (\mathtt{r}\hat{\ }2 + \mathtt{z}\hat{\ }2)\hat{\ }(-3/2)$$

for the expression. The effect will be to program the equivalent of equation (2.16). Similarly, right-click the Definitions in the Inverted Far Exterior branch of the model tree and select Variables. For the Geometric entity level menu choose Domain and for the Selection menu choose All domains. In the first line of the Variables table enter `PhiAnalytic` for the name and

$$-(1/2) * \mathtt{Q} * (\mathtt{a\_s}/\mathtt{a})\hat{\ }3 * \mathtt{z}$$

for the expression. The effect will be to program the equivalent of equation (2.20).

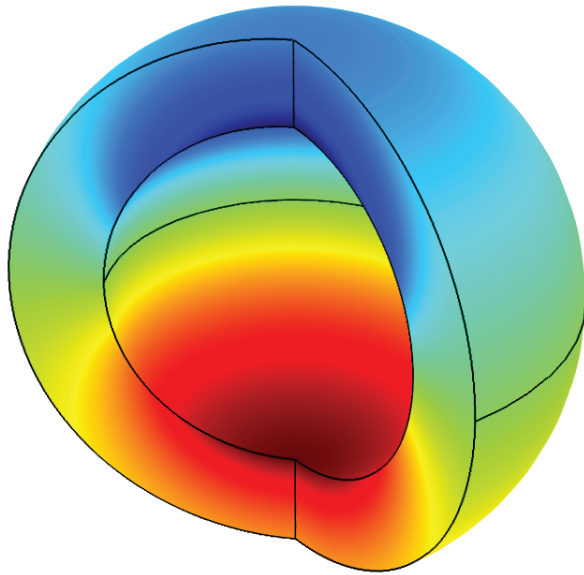## 4. Results for the velocity potential

Figure 4.1 Velocity potential, ϕ, in the Near Exterior by COMSOL. Here, and elsewhere, $a/a_s = 1.5$, the mesh is the default in both models, and the z-axis points up from the center. The ranges for ϕ are $(-0.5, 0.5)\,\mathrm{m^2/s}$ for the analytical solution and $(-0.498, 0.498)\,\mathrm{m^2/s}$ for the simulation.
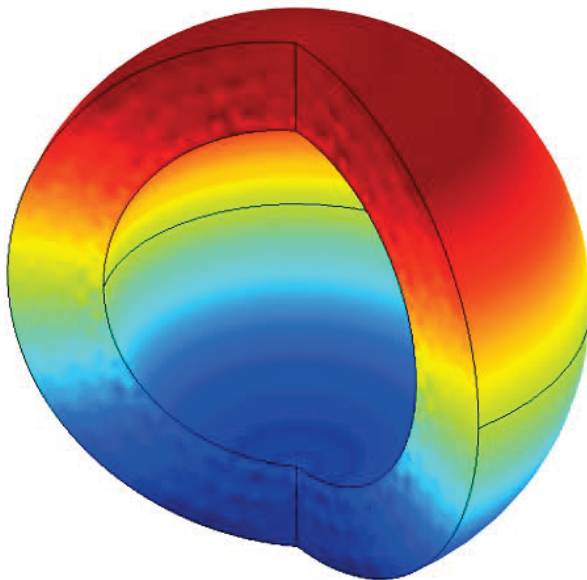

Figure 4.2 Discrepancy, $\Delta\phi := \phi - \phi_{\mathrm{analytic}}$, between COMSOL and analytical calculation in the Near Exterior. The range of $\Delta\phi$ is $(-2.63, 2.99) \times 10^{-3}\,\mathrm{m^2/s}$.
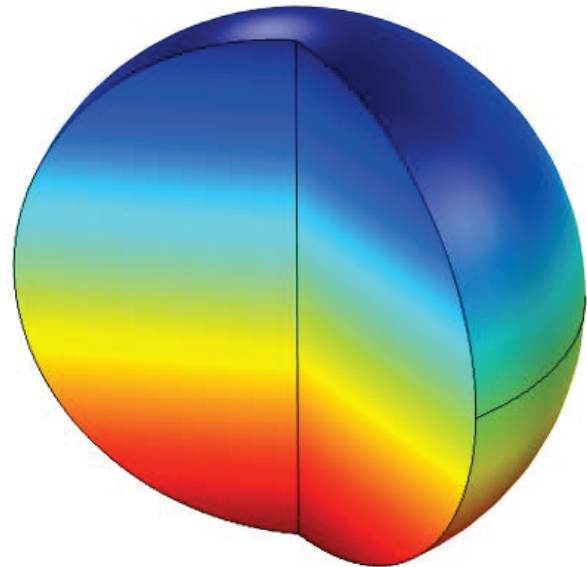

Figure 4.3 Velocity potential, Φ, in the Inverted Far Exterior by COMSOL. The range of Φ covers the intervals $(-0.222, 0.222)\,\mathrm{m^2/s}$ and $(-0.220, 0.220)\,\mathrm{m^2/s}$ for the (analytical, simulatied) solutions, respectively. The linear dependence upon z is in accord with equation (2.20).

5. Conclusions

C1. KELVIN's Inversion Theorem enables replacement of one flow in an unbounded domain by two flows, each of which is in a bounded domain;

C2. COMSOL enables simultaneous solution in the two bounded domains.

6. References

1. KELVIN, LORD Extrait d'une lettre de M. William Thomson à M. Liouville. *Journal de Mathématique Pures et Appliquées*, **10** (1845). See also *Reprint of Papers on Electrostatics and Magnetism by Sir William Thomson*, second edition, Cambridge, pages 144–146 (1884).

2. KELVIN, LORD Extrait de deux lettres adressées à M. Liouville par M. William Thomson. *Journal de Mathématique Pures et Appliquées*, **12** (1847). See also *Reprint of Papers on Electrostatics and Magnetism by Sir William Thomson*, second edition, Cambridge, pages 146–154 (1884).