Simulation Of A Vortex Ring: Dealing With The Unbounded, Doubly Connected Domain

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Abstract:

The present COMSOL model simulates the velocity field, $\mathbf{u}$, of a fluid flow subject to the following conditions: (1), the divergence of $\mathbf{u}$ is zero everywhere; (2), the curl of $\mathbf{u}$ is nonzero only in the interior of a torus of circular cross section, (in which region the only nontrivial component is the azimutal one, ω_ϕ, which, in turn is directly proportional to the distance, r, from the axis of symmetry); (3), there is no exterior boundary of the fluid; (4), the motion is independent of time when viewed by an observer who moves with the torus; and (5), relative to the same observer the normal component of $\mathbf{u}$ is continuous across the torus. The results enable computation of the propagation speed of the ring but show a non-physical discontinuity of the tangential component of $\mathbf{u}$ across the torus even if the circulations calculated about the inside and outside faces of the torus are equal.

Keywords: Unbounded domains, Vortex ring, Kelvin Inversion

1. Objective

The objective of the work reported herein was to develop an equation-based COMSOL model that might serve as a benchmark for fluid-flow problems having the following complications:

i. The region occupied by the fluid is unbounded;
ii. There is subregion in which the motion is rotational (*i.e.* the velocity field, $\mathbf{u}$, satisfies $\mathrm{curl}\,(\mathbf{u}) \neq \mathbf{0}$);
iii. There is a *doubly connected* subregion in which the motion is irrotational, *i.e.* $\mathrm{curl}\,(\mathbf{u}) = \mathbf{0}$.

2. A pair of second-order PDEs that replace a pair of first order PDEs

Let (x, y, z) be cartesian coordinates and let $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ be the corresponding right-handed orthonormal triad of vectors in the directions of the positive coordinate axes. I assume, here and elsewhere, that the velocity field is axysymmetric and without swirl. Let the z-axis coincides with the axis of symmetry and let (r, ϕ, z) be cylinderical coordinates, in which z has the same meaning as in the cartesian system and r and ϕ are related to x and y by

$$x = r\cos\phi \quad , \quad y = r\sin\phi \; . \qquad (2.1)_{1,2}$$

Let $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{k}}\}$ be the right-handed orthonormal triad of vectors associated with this system, in which $\hat{\mathbf{k}}$ has the same meaning as in the cartesian system and $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\phi$ are in the directions of increasing r and ϕ, respectively. Here, and elsewhere, I will refer to r and ϕ as the *transverse coordinate* and *azimuth angle*, respectively, and will apply the adjective *merdional* to any vector component or section of a geometric figure cut by half-plane of constant ϕ.

Let $\mathcal{R}$ denote the whole region of unbounded physical space and let $\mathcal{R}^c \subset \mathcal{R}$ denote the subregion of rotational motion occupied by the vortex core. I restrict attention to the case when the motion in $\mathcal{R}\backslash\mathcal{R}^c$ (*i.e.* the complement of $\mathcal{R}^c$ in the parent set $\mathcal{R}$) is irrotational. Specifically, I assume that the velocity field satisfies

$$\mathrm{div}\,\mathbf{u} = 0 \quad , \quad \mathrm{curl}\,\mathbf{u} = Ar \; , \qquad (2.2)_{1,2}$$

in which A is a piecewise uniform with a nonzero value only in $\mathcal{R}^c$. The solenoidality condition $(2.2)_1$ applies to an incompressible fluid. STOKES showed that one can represent any axisymmetric solenoidal motion by

$$\mathbf{u} = (1/r)\,\hat{\mathbf{e}}_\phi \times \nabla\psi \; , \qquad (2.3)$$

in which ψ is a differentiable scalar field. When $\mathbf{u}$ is a given velocity field ψ is called the *Stokes Stream Function*. Typically, as here, $\psi = 0$ on $r = 0$. Here the velocity field is meridional and has an expansion of the form

$$\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_z\hat{\mathbf{k}} \; , \qquad (2.4)$$

which defines u_r and u_z. Axisymmetry implies, moreover, that the derivatives of u_r and u_z with respect to azimuth angle vanish, *i.e.*

$$\partial u_r/\partial\phi = 0 \quad , \quad \partial u_z/\partial\phi = 0 \qquad (2.5)_{1,2}$$

When one substitutes the special forms (2.4) and (2.5) into the general formula for the expansion of the curl of a vector in a cylindrical coordinate system one finds that there is only one nontrivial term, namely the azimuthal one, *i.e.*

$$\operatorname{curl}\mathbf{u} = \omega_\phi \hat{\mathbf{e}}_\phi \ , \tag{2.6}$$

in which

$$\omega_\phi := \partial u_r/\partial z - \partial u_z/\partial r \ . \tag{2.7}$$

In view of (2.6) equation $(2.2)_2$ is equivalent to $\omega_\phi \hat{\mathbf{e}}_\phi = Ar\hat{\mathbf{e}}_\phi$, or

$$(\omega_\phi - Ar)\hat{\mathbf{e}}_\phi = \mathbf{0} \ . \tag{2.8}$$

Now COMSOL solves partial differential equations of second order with respect to the space derivatives. Equations $(2.2)_1$ & (2.8) are of first order and therefore not immediately suitable for substitution in the input fields of COMSOL's General Form PDE physics interface, for example. One may, however, derive a boundary-value problem, including a second order partial differential equation for $\mathbf{u}$, whose solutions satisfy $(2.2)_1$ and (2.8) by construction. To this end note that if $(2.2)_1$ and (2.8) both hold then so does the second order equation

$$\nabla(2\operatorname{div}\mathbf{u}) - \operatorname{curl}\left[(\omega_\phi - Ar)\hat{\mathbf{e}}_\phi\right] = \mathbf{0} \ . \tag{2.9}$$

If one expands (2.9) into components relative to the system $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{k}}\}$ and applies the axisymmetry assumption, one gets

$$-\hat{\mathbf{e}}_r\left\{\frac{\partial(-1)(\omega_\phi - Ar)}{\partial z} - \frac{\partial(2\operatorname{div}\mathbf{u})}{\partial r}\right\} + \hat{\mathbf{k}}\left\{\frac{1}{r}\frac{\partial}{\partial r}[r(-1)(\omega_\phi - Ar)] + \frac{\partial(2\operatorname{div}\mathbf{u})}{\partial z}\right\} = \mathbf{0} \tag{2.11}$$

after some rearrangement. Since the vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{k}}$ are linearly independent equation (2.11) implies that their coefficients vanish separately. Thus,

$$\frac{\partial(-1)(\omega_\phi - Ar)}{\partial z} - \frac{\partial(2\operatorname{div}\mathbf{u})}{\partial r} = 0 \tag{2.12}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}[r(-1)(\omega_\phi - Ar)] + \frac{\partial(2\operatorname{div}\mathbf{u})}{\partial z} = 0 \ . \tag{2.13}$$

If one multiplies (2.12) by -1 one may write the result in an equivalent matrix form, namely

$$\begin{pmatrix} \partial/\partial r & \partial/\partial z \end{pmatrix}\begin{pmatrix} 2\operatorname{div}\mathbf{u} \\ (\omega_\phi - Ar) \end{pmatrix} = 0 \tag{2.14}$$

with nominal dependent variable u_r. Alternatively, if one multiplies (2.13) by r one may write the result in an equivalent matrix form, namely

$$\begin{pmatrix} \partial/\partial r & \partial/\partial z \end{pmatrix}\begin{pmatrix} -r(\omega_\phi - Ar) \\ r\,2\operatorname{div}\mathbf{u} \end{pmatrix} = 0 \tag{2.15}$$

with nominal dependent variable u_z. If one regards $\operatorname{div}\mathbf{u}$ as an abbreviation for the right member of

$$\operatorname{div}\mathbf{u} = u_r/r + \partial u_r/\partial r + \partial u_r/\partial z \tag{2.16}$$

and ω_ϕ as an abbreviation for the right member of (2.7) then equations (2.14) and (2.15) are suitable for immediate substitution into the input fields of COMSOL's General Form PDE physics interface.

One may regard the system (2.14) and (2.15) as *necessary* conditions for the simulation of the original first order system $(2.2)_1$ and (2.8). I now turn to the question of *sufficiency*. Now (2.14) is equivalent to (2.12) and if one multiplies the latter by $\hat{\mathbf{e}}_\phi$ one gets

$$\operatorname{curl}\left[-(\omega_\phi - Ar)\hat{\mathbf{e}}_r + (2\operatorname{div}\mathbf{u})\hat{\mathbf{k}}\right] = \mathbf{0} \ . \tag{2.19}$$

In the mean time (2.15) is equivalent to (2.13) and one may write the latter in the form

$$\operatorname{div}\left[-(\omega_\phi - Ar)\hat{\mathbf{e}}_r + (2\operatorname{div}\mathbf{u})\hat{\mathbf{k}}\right] = 0 \ . \tag{2.20}$$

The general solution of (2.19) is

$$-(\omega_\phi - Ar)\hat{\mathbf{e}}_r + (2\operatorname{div}\mathbf{u})\hat{\mathbf{k}} = \nabla\Phi \ , \tag{2.21}$$

in which Φ is an arbitrary differentiable scalar field. If one substitutes (2.21) into (2.20) one gets

$$\operatorname{div}(\nabla\Phi) = 0 \ , \tag{2.22}$$

i.e. Φ satisfies LAPLACE's equation. According to classical potential theory (*e.g.* Ref. 2) if a scalar field, g, satisfies LAPLACE's equation in a simply connected domain and $\nabla g \bullet \hat{\mathbf{n}} = 0$ over the whole boundary (*i.e.* g satisfies homogeneous NEUMANN conditions) then $\nabla g = \mathbf{0}$ thoughout the domain. In the present example (2.22) shows that Φ satisfies LAPLACE's equation and according to (2.21) a sufficient condition for Φ to satisfy $\nabla\Phi \bullet \hat{\mathbf{n}} = 0$ (*i.e.* homogeneous NEUMANN conditions) is for the velocity field to satisfy

$$[-(\omega_\phi - Ar)\hat{\mathbf{e}}_r + (2\mathrm{div}\,\mathbf{u})\hat{\mathbf{k}}] \bullet \hat{\mathbf{n}} = 0\ . \qquad (2.23)$$

But (2.23) will hold any time the original first-order system $(2.2)_1$ & (2.8) holds uniformly over the boundary. In summary, if the system of partial differential equations (2.14) & (2.15) holds in a simply connected domain and the original first-order system $(2.2)_1$ & (2.8) holds uniformly over its boundary then the original first-order system must hold throughout the interior of the domain, thereby confirming the sufficiency to which I alluded earlier.

3. Conditions for uniqueness of u

Recall that $\mathcal{R}^c$ is the axysymmetric three-dimensional region of rotational motion and $\mathcal{R}\backslash\mathcal{R}^c$ is the complimentary axisymmetric region where the motion is irrotational. Here, and elsewhere, I will replace the symbol $\mathcal{R}$ by $\mathcal{D}$ in reference to the *two-dimensional meridonal section* of the corresponding *three dimensional axisymmetric region.* Thus $\mathcal{D}$ is the meridional section of $\mathcal{R}$, $\mathcal{D}^c$ is the meridional section of $\mathcal{R}^c$, $\mathcal{D}\backslash\mathcal{D}^c$ is the meridional section of $\mathcal{R}\backslash\mathcal{R}^c$, *etc.*

I assume that the $\mathcal{D}^c$ is a circular disk of radius c, whose center is situated at a distance r_m from the axis of symmetry and I assume that the plane $z = 0$ coincides with the equatorial plane of $\mathcal{R}^c$. I apply the adjectives *up* and *down* to the direction of $\hat{\mathbf{k}}$ and $-\hat{\mathbf{k}}$, respectively, and refer to the region $z > 0$ and $z < 0$ as the upper and lower halves of $\mathcal{R}$, respectively.

Under the foregoing assumptions one may expect u_r and u_z to be odd and even functions of z, respectively. There is therefore no need to simulate the velocity field in both the upper and lower halves of $\mathcal{R}$. Accordingly the present work reports results only for the upper half. Thus I will write $\mathcal{D}_+$ to denote the upper half of $\mathcal{D}$, $\mathcal{D}^c_+$ the upper half of $\mathcal{D}^c$, $\mathcal{D}_+\backslash\mathcal{D}^c_+$ the upper half of $\mathcal{D}\backslash\mathcal{D}^c$, *etc.*.

In the last section I appealed to classical potential theory in the assertion of conditons that ensure the vanishing of the right member of (2.21). Owing to space limitations, I will again appeal to classical potential theory, this time in the assertion of conditions for the vanishing of the vector difference $\mathbf{u}_2 - \mathbf{u}_1$ between two solutions, $\mathbf{u}_1$ and $\mathbf{u}_2$ of the same boundary-value problem. To this end note that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are both solutions of the original first order system $(2.2)_{1,2}$ then we have

$$\mathrm{div}\,(\mathbf{u}_2 - \mathbf{u}_1) = 0 \quad , \quad \mathrm{curl}\,(\mathbf{u}_2 - \mathbf{u}_1) = \mathbf{0}\ . \quad (3.1)_{1,2}$$

Suppose, now, that $(3.1)_{1.2}$ hold in a generic meridional section $\mathcal{D}^1_+$. Note that the boundary $\partial\mathcal{D}^1_+$ is a closed loop. Let $\hat{\mathbf{t}}$ denote the unit vector tangent to the loop $\partial\mathcal{D}^1_+$. For definiteness let $\hat{\mathbf{t}}$ be oriented in the right-handed sense relative to $\hat{\mathbf{e}}_\phi$.

On a part $\mathcal{P}^n \in \partial\mathcal{D}^1_+$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy a common boundary condition in which $\mathbf{u} \bullet \hat{\mathbf{n}}$ is given, we have

$$(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{n}} = 0\ . \qquad (3.2)$$

Alternatively, on a part $\mathcal{P}^t \in \partial\mathcal{D}^1_+$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy a common boundary condition in which $\mathbf{u} \bullet \hat{\mathbf{t}}$ is given, we have

$$(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{t}} = 0 \qquad (3.3)$$

In the present investigation I assume that $\mathbf{u} \bullet \hat{\mathbf{n}}$ and $\mathbf{u} \bullet \hat{\mathbf{t}}$ are given on adjoining but non-overlapping subsets of $\partial\mathcal{D}^1_+$ whose union is all of $\partial\mathcal{D}^1_+$. One may, accordingly, write $\partial\mathcal{D}^1_+ = \Sigma\mathcal{P}^n \cup \Sigma\mathcal{P}^t$, in which the summation signs acknowledge the possibility that either the subset of $\partial\mathcal{D}^1_+$in which $\mathbf{u} \bullet \hat{\mathbf{n}}$ is given or the one in which $\mathbf{u} \bullet \hat{\mathbf{t}}$ is given, or both, might be composed of disjoint parts.

According to classical potential theory the foregoing equations and definitions enable one to prove the following lemmas, which I assert here without proof:

Lemma 1. If $\mathbf{u}_2 - \mathbf{u}_1$ is a solution of the system $(3.1)_{1,2}$ over a meridional section $\mathcal{D}^1_+$ and if $\partial\mathcal{D}^1_+$ can be partitioned into two piecewise connected subsets that adjoin each other but do not overlap and such that $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{n}} = 0$ over one subset and $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{t}} = 0$ over the other then $\mathbf{u}_2 - \mathbf{u}_1 = \mathbf{0}$ uniformly over $\mathcal{D}^1_+$.

Lemma 2. Suppose $\mathbf{u}_2 - \mathbf{u}_1$ is a solution of the system $(3.1)_{1,2}$ over a meridional section $\mathcal{D}^1_+$ and suppose $\partial\mathcal{D}^1_+$ can be partitioned into four piecewise connected subsets that adjoin each other but do not overlap and such that $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{n}} = 0$ over two of these subsets and $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{t}} = 0$ over the other two. Let $\mathcal{C}$ be a simple oriented contour in $\mathcal{D}^1_+$ that starts on one of the two subsets where $(\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{t}} = 0$ and ends on the other and suppose that

$$\int_{\mathcal{C}} (\mathbf{u}_2 - \mathbf{u}_1) \bullet \hat{\mathbf{t}}\, ds = 0 \,, \tag{3.4}$$

in which s is an arc-length parameter that increases in the direction of $\hat{\mathbf{t}}$. Then $\mathbf{u}_2 - \mathbf{u}_1 = \mathbf{0}$ uniformly over $\mathcal{D}^1_+$.

Lemma 1 applies, in particular, to $\mathcal{D}^c_+$, *i.e.* the region occupied by the cross section of the half core in the region $z > 0$. Here $\mathbf{u} \bullet \hat{\mathbf{n}}$ is given (*via* an impermeability condition) on the semicircular edge of $\partial\mathcal{D}^c_+$ in the region $z > 0$ and $\mathbf{u} \bullet \hat{\mathbf{t}}$ is zero on the equatorial edge of $\partial\mathcal{D}^c_+$ (since $\mathbf{u} \bullet \hat{\mathbf{t}} = u_r$ there and u_r is odd in z).

Lemma 2 applies, in particualar, in $\mathcal{D}+\backslash\mathcal{D}^c_+$ (the meridional cut of the uper region of irrotational motion). Thus $\mathbf{u} \bullet \hat{\mathbf{n}}$ is given (*via* an impermeability condition) on the semicircular edge of $\partial(\mathcal{D}+\backslash\mathcal{D}^c_+)$ in the region $z > 0$ and on the centerline where $\mathbf{u} \bullet \hat{\mathbf{n}} = u_r = 0$. Moreover $\mathbf{u} \bullet \hat{\mathbf{t}}$ is zero on the disjoint equatorial edges of $\partial(\mathcal{D}+\backslash\mathcal{D}^c_+)$ (one inboard and one outboard) since $\mathbf{u} \bullet \hat{\mathbf{t}} = u_r$ there and u_r is odd in z.

Note that (3.4) will hold if $\mathbf{u}_1$ and $\mathbf{u}_2$ are subject to a subsidiary condition of the form

$$\int_{\mathcal{C}} \mathbf{u} \bullet \hat{\mathbf{t}}\, ds := \Gamma/2 \tag{3.5}$$

with given Γ. Here, Γ is the *circulation* about the core, *i.e.* the value that would result if $\mathcal{C}$ in (3.5) were a closed loop that embraced the whole core. The need for the circulation condition is typical in problems in doubly connected domains—as the present one would be if the domain of irrotational motion included both the lower and upper regions.

4. Simulation of u the unbounded exterior

Let (R, θ, ϕ) be spherical coordinates related to the cylindrical coordinates (r, ϕ, z) by

$$r = R \sin\theta \quad , \quad z = R\cos\theta \,, \qquad (4.1)_{1.2}$$

in which the azimuth angle ϕ has the same meaning as in the cylindrical system. I will refer to R and θ as the *radial* and *colatitudinal* coordinate, respectively. Let $\{\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ denote the right-handed orhonormal triad of vectors in the directions of increasing, R, θ, and ϕ, respectively, in which $\hat{\mathbf{e}}_\phi$ has the same meaning as in the cylindrical system.

Let $\mathbf{R}$ denote the position of a typical point in the physical domain $\mathcal{R}$ relative to the a fixed origin. One may construct a change of variable $\mathbf{R} \to \mathbf{q}$ that maps an unbounded subregion $\mathcal{R}^e \in \mathcal{R}$ (the superscript denotes *exterior*) to a bounded image $\mathcal{Q}$, which I will call a *proxy domain.* Before giving equations that define $\mathbf{R} \to \mathbf{q}$ I note that one may cover $\mathcal{Q}$ with a system of spherical coordinate (q, ϑ, φ) analogous to the one that covers the physical domain $\mathcal{R}$. To this end let $\{\hat{\mathbf{u}}_q, \hat{\mathbf{u}}_\vartheta, \hat{\mathbf{u}}_\varphi\}$ denote the right-handed orthonormal triad of vectors in the direction of increasing q, ϑ, and φ, respectively.

Let a denote a given constant length. One may now define $\mathbf{R} \to \mathbf{q}$ by the following rules:

$$Rq := a^2 \quad , \quad \theta := \vartheta \quad , \quad \varphi := \phi \,, \qquad (4.2)_{1,2,3}$$

$$\hat{\mathbf{u}}_q := \hat{\mathbf{e}}_R \quad , \quad \hat{\mathbf{u}}_\vartheta := \hat{\mathbf{e}}_\theta \quad , \quad \hat{\mathbf{u}}_\varphi := \hat{\mathbf{e}}_\phi \,. \qquad (4.2)_{4,5,6}$$

The system (4.2) implies that

$$R\frac{\partial}{\partial R} = -q\frac{\partial}{\partial q} \quad , \quad \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \vartheta} \quad , \quad \frac{\partial}{\partial \phi} = \frac{\partial}{\partial \varphi} \,. \qquad (4.3)_{1,2,3}$$

The foregoing definition of $\mathbf{R} \to \mathbf{q}$ is the equivalent of one that LORD KELVIN introduced in 1845 (Ref. 1). Following custom (*e.g.* Ref. 2, pages 231–233) I will refer to this definition of $\mathbf{R} \to \mathbf{q}$ as *Kelvin Inversion.*

Now COMSOL's 2D axisymmetric algorithm requies the use of cylindrical coordnates. To this end let (ϖ, φ, ζ) denote cylindrical coordinates that cover $\mathcal{Q}$ and are related to the corresponding spherical coordinates (q, ϑ, φ) by

$$\varpi = q \sin\vartheta \quad , \quad \zeta = q\cos\vartheta \,, \qquad (4.4)_{1.2}$$

in which φ is the same azimuth angle as in the spherical system and let $\{\hat{\mathbf{u}}_\varpi, \hat{\mathbf{u}}_\varphi, \hat{\mathbf{u}}_\zeta\}$ denote the orthonormal triad of vectors in the directions of increasing ϖ, φ, and ζ, respectively.

The conditions I have given thus far enable the transformation of the second-order PDE for $\mathbf{u}$ in

$\mathcal{R}$, namely(2.9), to the corresponding second order PDE for $\mathbf{u}$ in $\mathcal{Q}$. In the interest of brevity I will compress the following derivation by presenting it in the form of a set of instructions followed by a description of what I get when I carry them out. Specifically,

(i) Expand (2.9) into components in spherical coordinates in $\mathcal{R}$;
(ii) Transform the result of (i) to the corresponding components in spherical coordinates in $\mathcal{Q}$ *via* (4.2) and (4.3);
(iii) Transform the result of (ii) from spherical coordinates in $\mathcal{Q}$ to cylindrical coordinates in $\mathcal{Q}$;
(iv) Express the two nontrivial components of the resulting vector PDE in matrix form analogous to the system (2.14), (2.15).

These operations yield

$$(\,\partial/\partial\varpi \quad \partial/\partial\zeta\,)\begin{pmatrix}\varpi\, F_{\varpi}\\ \varpi\, F_{\zeta}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\ , \qquad (4.5)$$

with nominal dependent variable u_{ϖ}, and

$$(\,\partial/\partial\varpi \quad \partial/\partial\zeta\,)\begin{pmatrix}F_{\zeta}\\ -F_{\varpi}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\ , \qquad (4.6)$$

with nominal dependent variable u_{ζ}, in which F_{ϖ} & F_{ζ} are abbreviations for the longer expressions

$$F_{\varpi} = 2(u_{\varpi}/\varpi)a^{-2}S + a^{-2}[2u_{\zeta,\zeta}S - (u_{\varpi,\zeta} + u_{\zeta,\varpi})C] \\ -2a^{-2}(S^2u_{\varpi,\varpi} + SCu_{\varpi,\zeta} + CSu_{\zeta,\varpi} + C^2u_{\zeta,\zeta})S \qquad (4.7)$$

and

$$F_{\zeta} = 2(u_{\varpi}/\varpi)a^{-2}C - a^{-2}[(u_{\varpi,\zeta} + u_{\zeta,\varpi})S - 2u_{\varpi,\varpi}C] \\ -2a^{-2}(S^2u_{\varpi,\varpi} + SCu_{\varpi,\zeta} + CSu_{\zeta,\varpi} + C^2u_{\zeta,\zeta})C\ . \qquad (4.8)$$

Here (u_{ϖ}, u_{ζ}) are defined such that $\mathbf{u} = u_{\varpi}\hat{\mathbf{u}}_{\varpi} + u_{\zeta}\hat{\mathbf{u}}_{\zeta}$, the commas denote partial differentiation, and

$$C := \zeta/q \quad , \quad S := \varpi/q \quad , \quad q = (\varphi^2 + \zeta^2)^{1/2}\ . \qquad (4.9)_{1,2,3}$$

Geometries in two components. The present model employs two components. In both components a is a the radius of sphere centered on the origin $(r, z) = (0, 0)$ and sized so that it intersects the toroidal boundary of the vortex core perpendicularly. Here, and elsewhere, I will denote this sphere by $\mathcal{S}_a$ and call it *the reflecting sphere*. For later reference let $\mathcal{S}_{2a}$ denote the sphere concentric with $\mathcal{S}_a$ but with twice the radius.

In Component 1 the geometry is an assembly of three sudomains, namely: $\mathcal{R}^c$ (the vortex core); $\mathcal{R}^i$ (the region of irrotational motion exterior to the core but interior to $\mathcal{S}_a$); and $\mathcal{R}^{ne}$ (the region exterior to $\mathcal{R}^c$ and $\mathcal{R}^i$ but interior to $\mathcal{S}_{2a}$). In each of the subdomains $\mathcal{R}^c$ and $\mathcal{R}^i$ COMSOL solves the system (2.14) and (2.15). In the subdomain $\mathcal{R}^{ne}$ COMSOL imports the solution from Component 2 by means of a model coupling operator of General Extrusion type.

In Component 2, the geometry is an assembly of two components, namely $\mathcal{Q}^{ne}$ (the image of $\mathcal{R}^{ne}$ under KELVIN Inversion) and $\mathcal{Q}^{fe}$ (the image under KELVIN Inversion of the exterior of $\mathcal{S}_{2a}$). The superscripts *ne* and *fe* stand for *near exterior* and *far exterior*, respectively. COMSOL solves the system (4.5) and (4.6) over the assembly consisting of $\mathcal{Q}^{ne}$ and $\mathcal{Q}^{fe}$. The interface between the two is an Identity Pair across whcih COMSOL applies a Continuity boundary condition.

Exchange of information between the components across a portal. I will use the term *portal* to describe the part of the reflecting sphere between the centerline and the boundary of the vortex core. The algorithms in the two Components trade information across the portal to ensure that the solution in $\mathcal{R}^{ne}$ (which is the KELVIN Inverse of the computed solution in $\mathcal{Q}^{ne}$) is a proper contination of the computed solution in $\mathcal{R}^i$. Thus Component 1 includes definitions of two model coupling operators of Boundary Similarity type, each of whose geometric scope is the portal: they differ in that the source for one is the destination for the other, the source and destination being the two sides of the portal. The physics interfaces that solve (2.14) and (2.15) have Dirichet conditions on the portal that import values of u_r and u_z from the other side (namely u_{ϖ} and u_{ζ}, respectively). The physics interfaces that solve (4.5) and (4.6) have Flux/Source boundary conditons that import values of the fluxes imported from the other side. To be specific the expressions that COMSOL requires in the field it labels "Boundary

Flux/Source" in the Flux/Source condition under General Form PDE physics interfaces for (4.5) and (4.6) are

$$-n_\varpi \varpi F_\varpi - n_\zeta \varpi F_\zeta \tag{4.10}$$

and

$$-n_\varpi F_\zeta - n_\zeta(-F_\varpi)\ , \tag{4.11}$$

respectively, in which F_ϖ and F_ζ have values imported from Component 1, namely

$$F_\varpi = a^{-4}(r^2+z^2)^{1/2}\{2r[(u_r/r)+u_{r,r}+u_{z,z}] + z(u_{r,z}-u_{z,r})\} \tag{4.12}$$

and

$$F_\zeta = a^{-4}(r^2+z^2)^{1/2}\{-r(u_{r,z}-u_{z,r}) + 2z[(u_r/r)+u_{r,r}+u_{z,z}]\}\ . \tag{4.13}$$

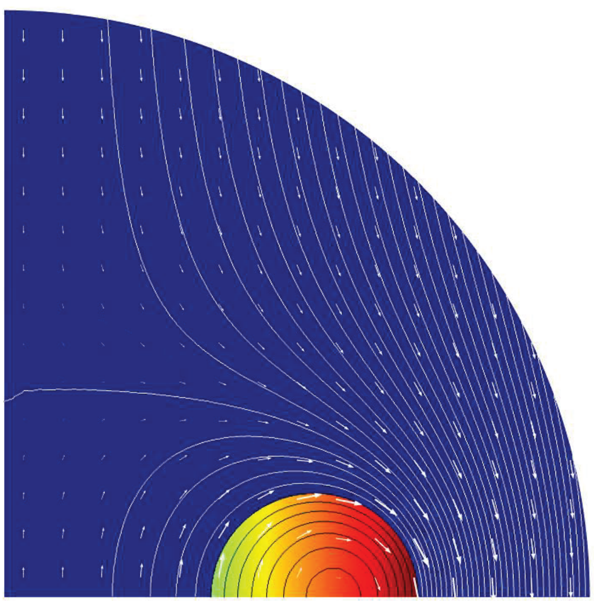

Figure 4.1 Meridional cut through a vortex ring as seen by an observer moving with the ring. Colors show azimuthal vorticity, solid lines are contours of STOKES stream function (in equal increments) and arrows show velocity vectors. Note the recirculation bubble that follows the ring. Here $r_m = 1[m]$, $\Gamma = 1[m^2/s]$, and $c/r_m = 1/3$. The computed value of W_∞ is -0.028327 $[m/s]$.

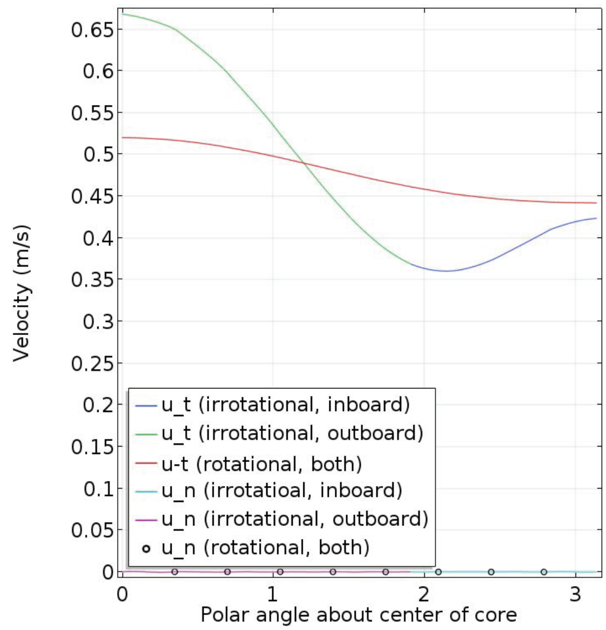

Figure 4.2 Tangential (u_t) and normal (u_n) components of velocity over the core boundary on its two sides, one where the motion is irrotational and the other where it is rotational.

Transformation of normal velocity under Kelvin Inversion. The near exterior $\mathcal{R}^{ne}$ has a face that abuts the vortex core along a collar-shaped surface one of whose edges is on the reflecting sphere while the other is on the outer equator of the core boundary. Since this collar is part of the core boundary the normal velocity $\mathbf{u}\bullet\hat{\mathbf{n}}$ is subject to an impermeabilty condition there. In the present model COMSOL calculates the velocity field in Ω^{ne} in Component 2 so one must reexpress the impermeability condition on this collar under KELVIN Inversion. The identity relevant to this purpose is

$$\mathbf{u}\bullet\hat{\mathbf{n}} = u_\zeta[n_\zeta - 2C(Cn_\zeta + Sn_\varpi)] + u_\varpi[n_\varpi - 2S(Cn_\zeta + Sn_\varpi] \tag{4.14}$$

On the propagation speed of the ring. In a reference frame moving with the ring the remote fluid is in motion. To this end one may include under the physics interface for equation (4.6) a Pointwise Constraint with a single point for its geometric scope that equates the expression $u_z - W_\infty$ to zero. Here W_∞ is an unknown dependent variable. To instruct

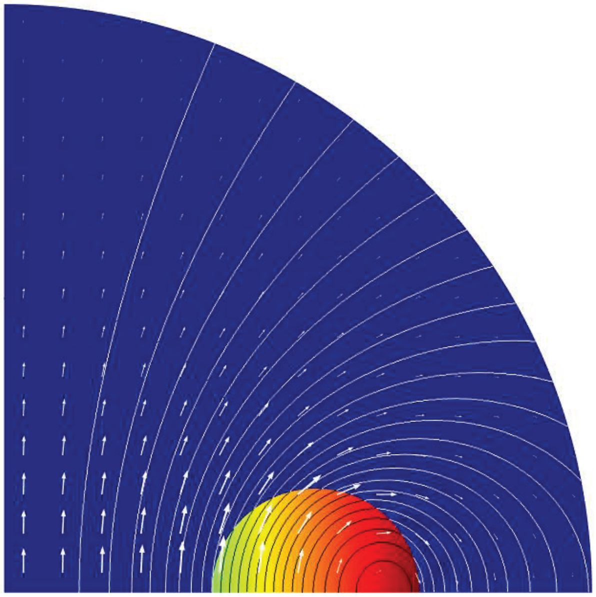

Figure 4.3 Same as Fig. 4.1 but this time the observer is at rest relative to the remote undisturbed fluid.

COMSOL to find its value one may insert a physics interface of Global ODEs and DAEs type in Component 2 that declares W_∞ as its dependent variable and specifies

$$\int_{\mathcal{C}} \mathbf{u} \bullet \hat{\mathbf{t}}\, ds - \Gamma/2 \tag{4.15}$$

as its constraint expression [which will vanish if the circulation condition (3.5) holds]. The value of Γ is prescribed (with unit value) in the global parameters list. The contributions to the integral in (4.15) from the parts of $\mathcal{C}$ interior and exterior to $\mathcal{S}_a$ are calcuated separately, the latter in Component 2 *via* the identity

$$\begin{aligned}\mathbf{u} \bullet \hat{\mathbf{t}}\, ds =&(a^2/q^2)[(u_\zeta t_\zeta + u_\varpi t_\varpi)\, d\sigma \\ &- 2(u_\zeta C + u_\varpi S)(Ct_\zeta + St_\varpi)\, d\sigma\ , \end{aligned}\tag{4.16}$$

in which $d\sigma := \|d\mathbf{q}\|$. The definitions of r_m, c, and Γ imply that A in $(2.2)_2$ equals $\Gamma/(r_m \pi c^2)$. See Figs. 4.1 & 4.2 for sample results in a reference frame that moves with the vortex ring.

In the case when the reference frame at rest relative to the remote fluid u_z is set to zero at infinity. The impermeability condition on the core boundary involves an unknown rise velocity W_{rise} for that boundary, which then becomes the dependent variable to be found by the the Global ODEs and DAEs physics interface (in which the constraint is the circulation condition, as before). See Figure 4.3 for sample results in this frame.

5. Discussion

Owing to the discontinuity in the tangential velocity across the boundary of the core the present results are incompatible with continuity of pressure. One conjecture is that there exists a solution that has a *non*circular cross section but is otherwise compatible with the present model, including circulation about the core, the area and centroid of the cross section of the core, and the same dependence of ω_ϕ with respect to r. An investigation to find the shape of the cross section, if it exists, *via* tools in COMSOL's optimzation module presents an opportunity for further development.

6. Conclusions

1. COMSOL enables computation of a solenoidal velocity field in a physical domain $\mathcal{R}^i \cup \mathcal{R}^e$—in which $\mathcal{R}^i$ and $\mathcal{R}^e$ denote a bounded interior and an unbounded exterior regions, respectively—by simultaneous solution for the flows in $\mathcal{R}^i$ and $\mathcal{Q}$, in which $\mathcal{Q}$ is a *bounded proxy* for $\mathcal{R}^e$;
2. COMSOL's General Extrusion model coupling operator enables one to give effect to the change of independent variable (Kelvin Inversion) that maps $\mathcal{R}^e$ to $\mathcal{Q}$;
3. The simultaneous assumptions that the vortex core has circular cross section and that the azimuthal vorticty in the core is directly proportional to the distance from the centerline enables one to satisfy continuity of normal but not tangential velocity across the core boundary.

7. References

1. Kelvin, Lord Extrait d'une lettre de M. William Thomson à M. Liouville. *Journal de Mathématique Pures et Appliquées*, **10**, 1845. [In *Reprint of Papers on Electrostatics and Magnetism by Sir William Thomson*, second edition, Cambridge, 1884, pages 144–146.]

2. Kellogg, O.D. *Foundations of Potential Theory*, Springer, 1929.