Implementation of the Perfectly Matched Layer to Determine the Quality Factor of Axisymmetric Resonators in COMSOL

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Abstract: Due to the inseparability of the wave equation, numerical methods are needed to develop an accurate electromagnetic model for various axisymmetric optical resonators such as micro-discs and micro-toroids. A COMSOL model for axisymmetric resonators has already been developed, however that model lacks the capability to determine the quality factor of a micro-cavity precisely. Here our purpose is the implementation of a perfectly matched layer to determine the quality factor of axisymmetric resonators with high accuracy in COMSOL.

Keywords: Perfectly matched layer (PML), Quality factor, Axisymmetric resonators

1 Introduction

Perfectly matched layers (PML) act as artificial boundaries to truncate the computation domain of open region scattering problems in the finite element method. The whispering gallery modes (WGM) of an open optical micro-cavity radiate into surroundings and a PML is required in order to block the unwanted reflections from the boundaries of the computation domain. One research group [1] has developed a COMSOL model for open axisymmetric resonators without applying any transverse approximation to the wave equation. However in their model no PML has been implemented and as a result the WGM quality factor can not be determined accurately. In that model, the quality factor due to the WGM radiation has been estimated by placing a bound on its minimum and maximum possible values. Determination of the quality factor with high accuracy is important in certain applications such as cavity ring down spectroscopy where decay time depends upon the quality factor. For accurate determination of the WGM quality factor we have implemented PML along the boundaries of the computation domain. We treat the PML as an anisotropic absorber and implement it in the cylindrical coordinate system. Our model is applicable to any axisymmetric resonator geometry but due to the availability of analytical expressions for spherical resonators, we have tested our model by determining the quality factors of a silica micro-sphere in air. We have found that our simulation results are in excellent agreement with the analytical results.

2 Mathematical Description

Applying Galerkin's method to the wave equation and after using the boundary conditions for open resonators, one can arrive at the FEM equation in the 'Weak Form' [1]:

$$\int_V \left((\vec{\nabla} \times \tilde{\vec{H}}^*)\epsilon^{-1}(\vec{\nabla} \times \vec{H}) - \alpha(\vec{\nabla} \cdot \tilde{\vec{H}}^*)(\vec{\nabla} \cdot \vec{H}) + c^{-2}\tilde{\vec{H}}^* \cdot \frac{\partial^2 \vec{H}}{\partial t^2} \right) dV \tag{1}$$

where $\vec{H}$ represents the magnetic field of the resonator and $\tilde{\vec{H}}$ represents the test magnetic field, an essential component of the weak form. The second term of the equation 1 represents a penalty term to suppress the false solutions, a well known problem in finite element formulations [2]. Because of geometry of the spherical cavity none of the field components will depend upon the azimuthal coordinate ϕ, resulting in reduction of the 3D problem to a 2D problem.

A PML can be treated as an anisotropic absorber [3] in which the diagonal permittivity and permeability tensors of the absorber are modified according to the equation 2.

$$\bar{\epsilon} = \epsilon\bar{\Lambda}, \bar{\mu} = \mu\bar{\Lambda}, \tag{2}$$

$\bar{\Lambda}$ is in cylindrical coordinates and is given by equation 3 [4]:

$$\bar{\Lambda} = \left(\frac{\tilde{r}}{r}\right)\left(\frac{s_z}{s_r}\right)\hat{r} + \left(\frac{r}{\tilde{r}}\right)(s_z s_r)\hat{\phi} + \left(\frac{\tilde{r}}{r}\right)\left(\frac{s_r}{s_z}\right)\hat{z} \tag{3}$$

where

$$s_r = \begin{cases} 0 & 0 \le r \le r_m \\ 1 - jG\left(\frac{r - r_m}{t_p}\right)^2 & r > r_m \end{cases}$$

$$s_z = \begin{cases} 1 - jG\left(\frac{z_{ml} - z}{t_{zl}}\right)^2 & z < z_{ml} \\ 1 & z_{ml} \le z \le z_{mu} \\ 1 - jG\left(\frac{z - z_{mu}}{t_{zu}}\right)^2 & z > z_{mu} \end{cases}$$

$$\tilde{r} = \begin{cases} r & 0 \le r \le r_m \\ r - jG\left(\frac{(r - r_m)^3}{3t_r^2}\right) & r > r_m \end{cases}$$

where t_r, t_{zu}, t_{zl} are the PML thicknesses in the radial, +z and -z directions respectively and r_m, z_{mu}, z_{ml} are the locations of the start of PML in the radial, +z and -z directions respectively. G is a real parameter whose optimum value lies between 5-6 for PML thickness of $\lambda/4$ and FEM mesh thickness of $\lambda/20$.

In order to incorporate the PML, equation 1 is rederived in the following form:

$$\int_V \left((\vec{\nabla} \times \vec{H}^*)\bar{\epsilon}^{-1}(\vec{\nabla} \times \vec{H}) - \alpha(\vec{\nabla} \cdot \vec{H}^*)(\vec{\nabla} \cdot \vec{H}) + c^{-2}\vec{H}^* \cdot \bar{\mu} \cdot \frac{\partial^2 \vec{H}}{\partial t^2} \right) dV \tag{4}$$

By casting equation 4 into COMSOL on the same lines as outlined in the [1], a full vectorial finite element model of a silica sphere in air can be obtained. Since the PML introduces losses in the computational domain, the resonant frequency (f_r) will be a complex number and the quality factor due to the WGM radiation can be calculated as:

$$Q_{wgm} = \frac{\Re(f_r)}{2\Im(f_r)} \tag{5}$$

3 Results

Figure 1 shows the fundamental TM mode of a silica spherical cavity in air. We plotted the

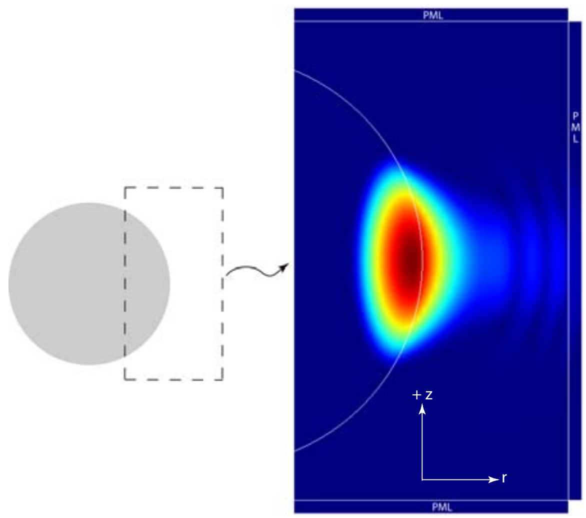

Figure 1: Fundamental TE mode of a $12\mu m$ silica micro-sphere in air (False Colors)

quality factor due to the TE/TM fundamental whispering gallery mode radiation for various sphere diameters at 850nm. Figures 2 and 3 show the comparison of COMSOL simulation results and results obtained by using analytical expressions [5] for a spherical cavity.

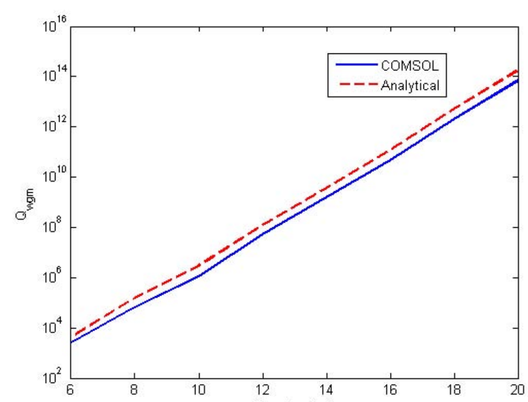

Figure 2: . Quality factor of fundamental TE modes at 850nm

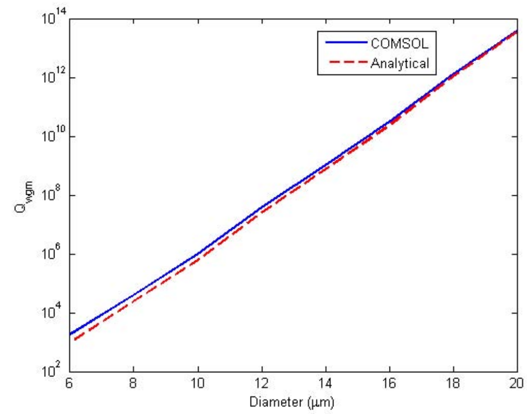

Figure 3: . Quality factor of fundamental TM modes at 850nm

4 Conclusions

The excellent agreement between the simulation and analytical results shows the correct working of PML. This model is also applicable to various micro-cavity geometries such as discs and toroids. Therefore, our model is an improved version of already introduced COMSOL model for axisymmetric resonators. The improved model will find wide usage in many micro-cavity research applications.

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6 References

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