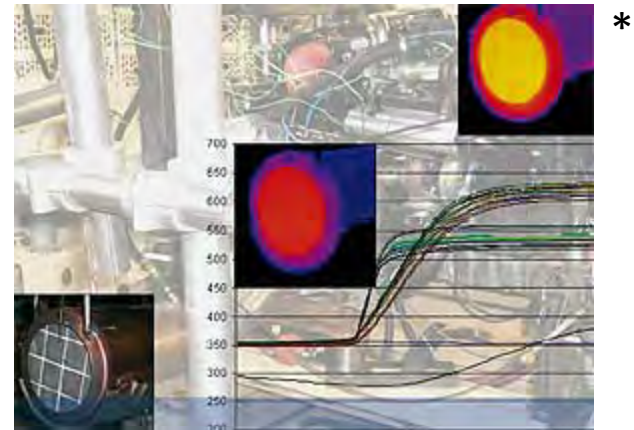
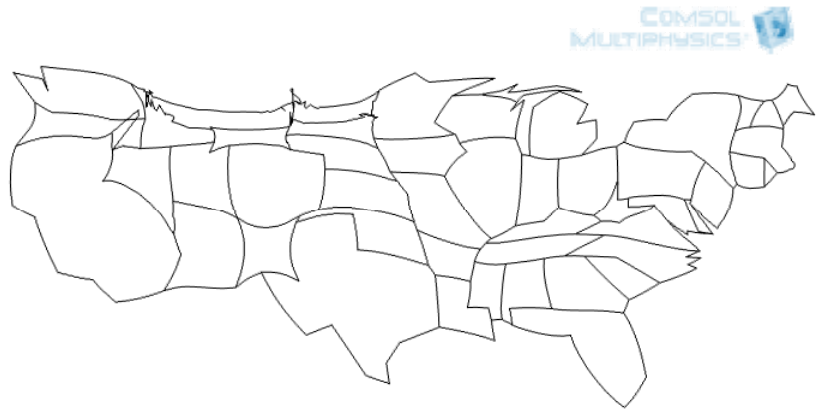


Making Cartograms and Using Them for Data Acquisition



“I bet you would not think COMSOL could do this, and that it is actually practical in engineering.”

P. Kip Mercure, The Dow Chemical Company, Midland, Michigan, USA

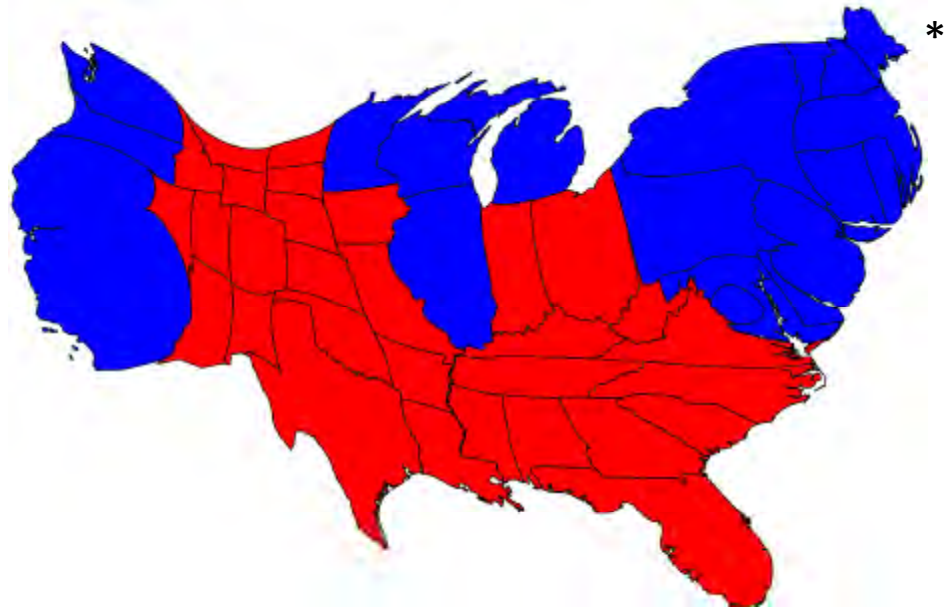
Robert P Haley, ATM Research, Midland, Michigan, USA

*C. Friedrich et al, “Diesel Vaporizer for Particulate Filter Regeneration”, MTZ worldwide Edition: 2009-05

Introduction to Cartograms:

- We can construct cartograms using a diffusion method in COMSOL Multiphysics.
- A cartogram is a geographical map distorted to represent some measure, for example population, while trying to keep the shape of regions recognizable.
- Making cartograms is a useful function, but not typically seen in engineering projects.

Typical cartogram: State area proportional to the number of electoral votes in the 2004 election



* <http://politicalmaps.org/2004-presidential-election-maps/>

Use of COMSOL Multiphysics #1:

- The diffusion-based method of Gastner and Newman* distorts the boundaries of a map to represent thematic data.
- A moving mesh distorts zones to distribute approximation error equally over the spatial region.
 - The diffusion calculation uses the “Mathematics Interface.” Thematic data, such as population, establishes the initial values for the diffusion problem. The gradient of the solution generates a velocity term for the movement of the boundaries.
 - The “Moving Mesh Interface” moves the boundaries according to the velocity function generated from the diffusion problem.

*Michael T. Gastner and M. E. J. Newman, “Diffusion-based method for producing density-equalizing maps”, Proceedings of the National Academy of Science, 101(20), pp 7499-7504 (May 18, 2004) doi10.1073pnas.0400280101

Diffusion-based method of Gastner & Newman

- Gastner & Newman start with the idea that if there is a non-equal distribution of the thematic variable (e.g. population density), diffusion would cause the distribution to become more even.
- If the boundaries “flow” with the population, the resulting boundary areas should reflect an equal population

Diffusion formulation

- Diffusion is described by the differential equation: $\frac{\partial p}{\partial t} = D \nabla^2 p$
 - The numerical value of the diffusion coefficient does not matter for the cartogram method, but keeping track of units may aid understanding
- The initial condition, $p[t=0]$, is the population density distribution
- If “insulating” boundary conditions are used, the total amount of population will remain constant within the geometry chosen
- It is useful to include non-populated areas such as lakes and oceans to aid in preserving a recognizable shape
 - These regions are specified with the same overall average population density as the populated region

Moving the boundaries

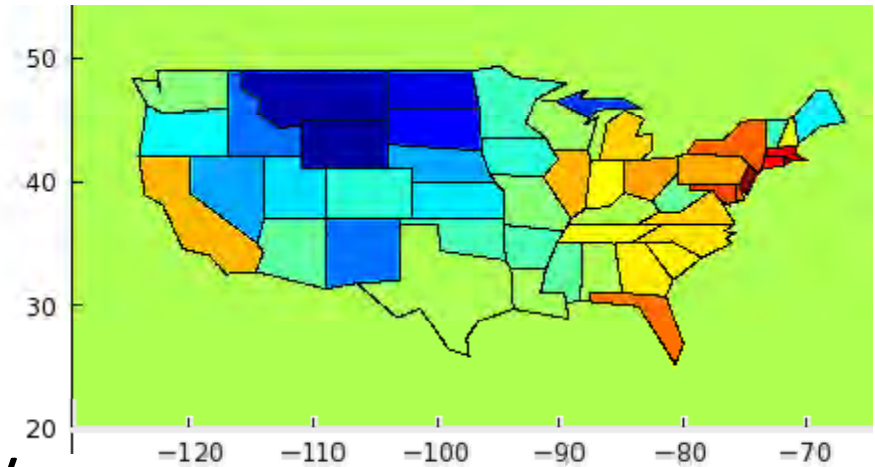
- Given the solution to the diffusion problem, $p[t,x,y]$, the flux is expressed as: $J = -D \nabla p$
- The flux can also be expressed as a concentration value times a velocity: $J = p v$
- The velocity used to move the boundaries is:

$$v = \frac{-D \nabla p}{p}$$

- Note that with the units for the diffusion coefficient of $\text{length}^2/\text{time}$, the velocity term will have units of $\text{length}/\text{time}$

Application to map

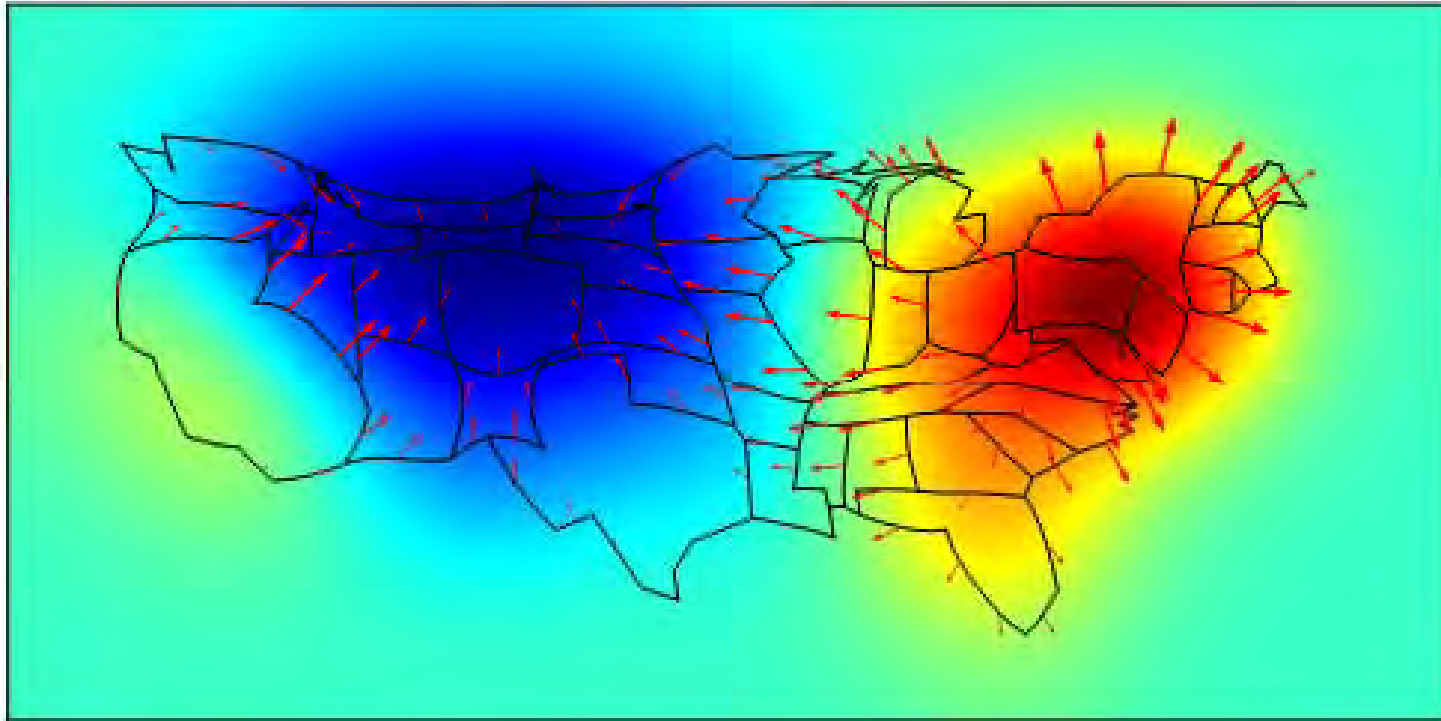
- Simplified state borders generated as .dxf files & imported into the geometry
- The initial values of each state are set to the population density
 - Color plotted is $\log(\text{population density})$
 - Factor of $\sim 5000x$ between extremes
 - Initial value of the surrounding region (lakes, oceans, Canada, Mexico) set to average population density of USA
- The population diffuses out



Implementation in Comsol

- Simplified state borders imported via .dxf
- Diffusion problem uses Mathematics General Form PDE
 - Initial values for each state are entered as the population density
 - The initial value for non-populated areas is entered as the average population density across all of the states
 - Zero flux boundary conditions are used on the outer rectangle
- Border movement calculated by Deformed Mesh
 - Outer rectangle border has prescribed mesh displacement of zero
 - A prescribed mesh velocity is used for the state borders
 - Because of sharp differences in population densities initially, the mesh velocity for border movement is damped at the beginning
 - $V_x = -u_y \cdot (1 - \exp(-10 \cdot t)) / u$
 - $V_y = -u_x \cdot (1 - \exp(-10 \cdot t)) / u$
- Damping Coefficient of 1 gives equilibrium in ~ 10 time units

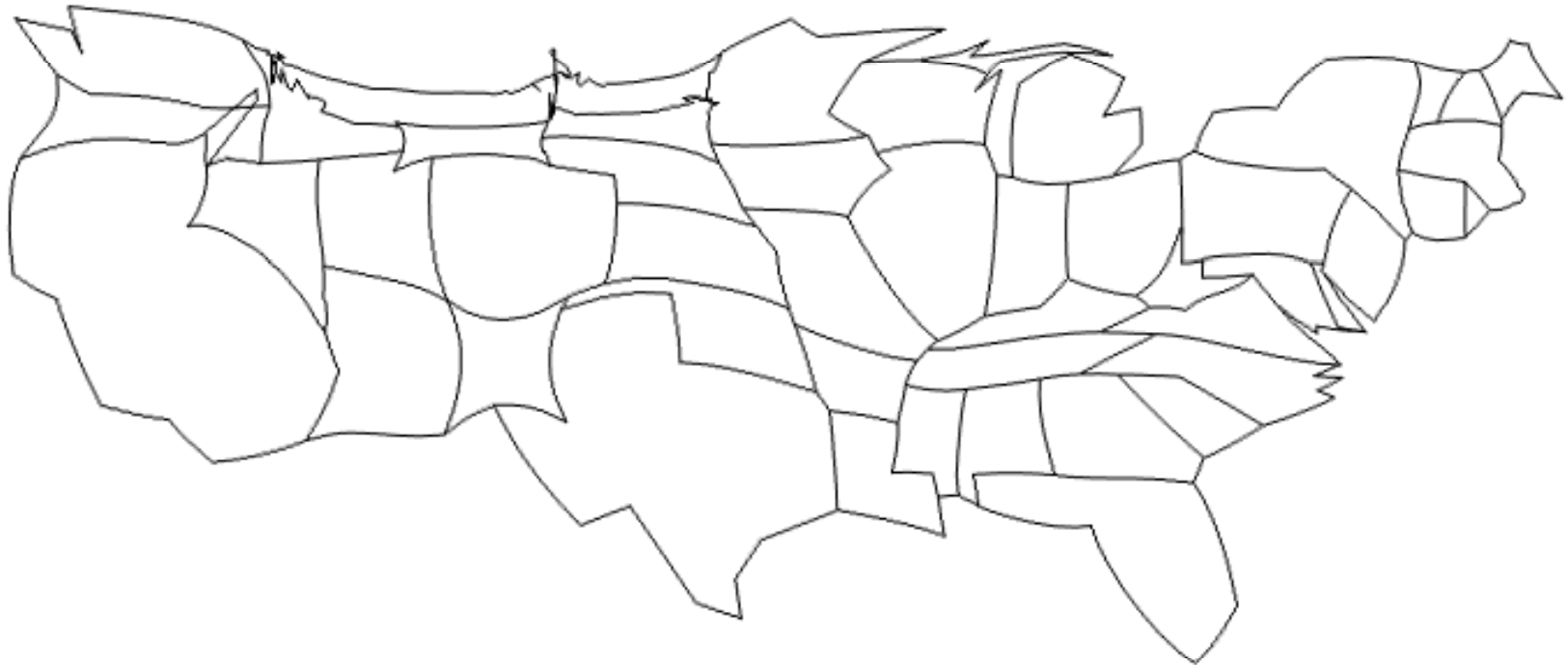
Diffusion process



- Early stage in the diffusion process
- Arrows show the direction of border motion
- The reddish areas of high population density are expanding, the bluish areas of low population density are contracting

Final result

CONSOL
MULTIPHASES

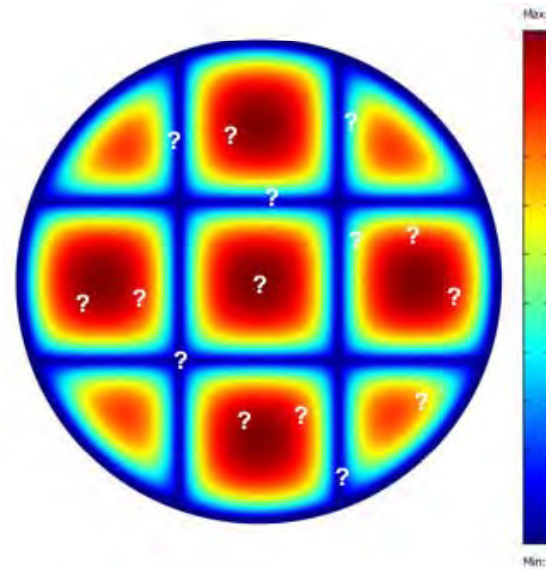


- Less than 0.9x difference in state population density

Introduction to Sensor Placement:

- The cartogram construction algorithm can be used to find an optimal placement of sensors to accurately reproduce spatial measurements such as temperature.
- The problem is to find the optimal placement of a small number of thermocouples to be able to reconstruct the full temperature profile developed during an experiment.

Typical temperature profile
for a thermal stress test:

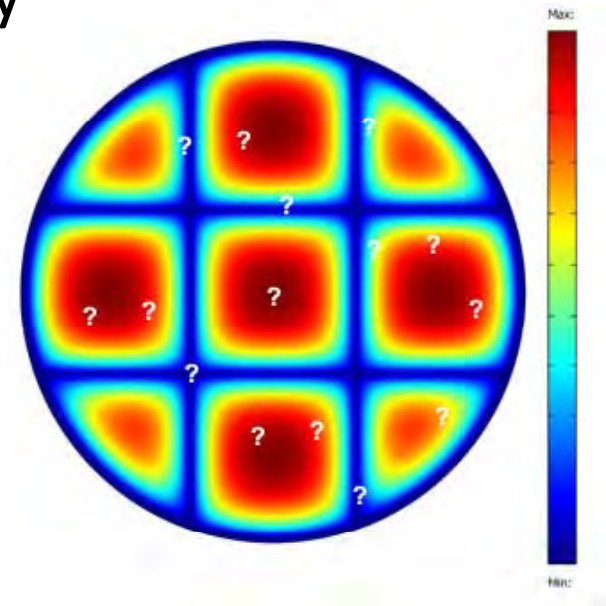


Use of COMSOL Multiphysics #2:

- Initial model of heat generation and conduction used to generate a solution for the system
- Heat transfer problem is a second order
 - reasonably approximated by cubic functions
 - error term in a cubic approximation function is on the order of the value of the second derivative of the solution value.
- COMSOL Multiphysics generates Laplacian of temperature used to initialize cartogram algorithm
- Uniform distribution of points applied to the cartogram, and locations mapped back to the original geometry.
 - This results in a distribution of measurement points that equally distributes the approximation error across the spatial region.

Temperature distribution in an approximate model

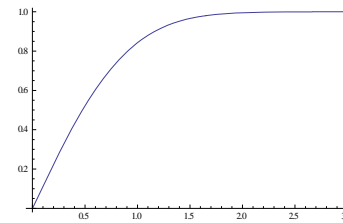
- Practical problem: Thermal shock testing of diesel particulate filters built up from segments
- Thermal model uses parameters close to actual values, heating rates are arbitrary
- The irregular boundaries make thermocouple placement non-intuitive
- Statement of problem:
“what is best placement of thermocouples to approximate the temperature profile?”



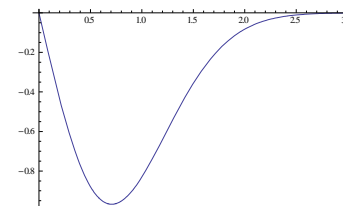
Even error distribution

- Carl de Boor shows that the approximation error for a spline is proportional to the second derivative
- For best sensor placement, the approximation error should be equi-distributed
- Consider the error estimate as a cartogram problem:
re-distribute the sensors to equalize the error distribution

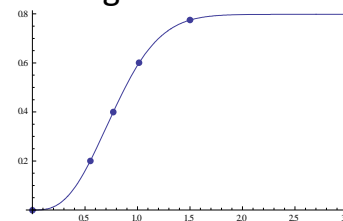
Function to be approximated



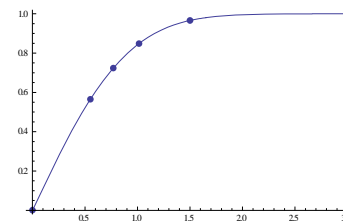
2nd derivative of function



Integral of square of 2nd derivative
with sample points placed evenly
along the cumulative value

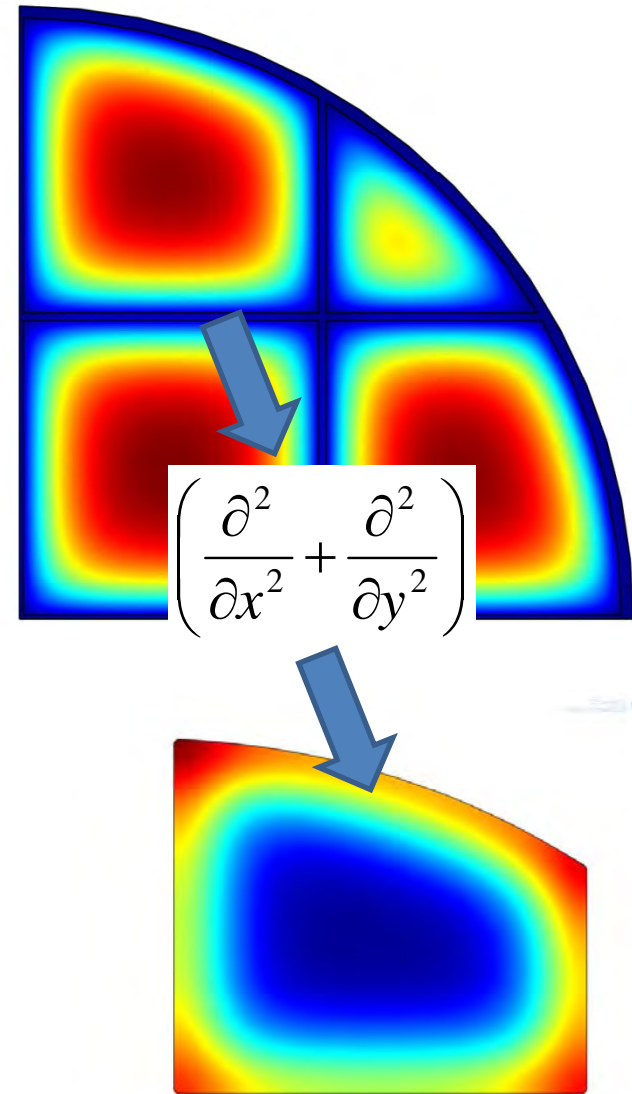


Sample points on original function



Procedure for redistribution

- Solve the heat conduction problem to find a typical temperature distribution
- Extract the Laplacian of the temperature data as an estimate of the approximation error
- Take the error estimate and use a diffusion process to equalize the error, producing a transform from the original coordinates to an “equal-error” coordinate system
- Use this transform to redistribute sensor locations

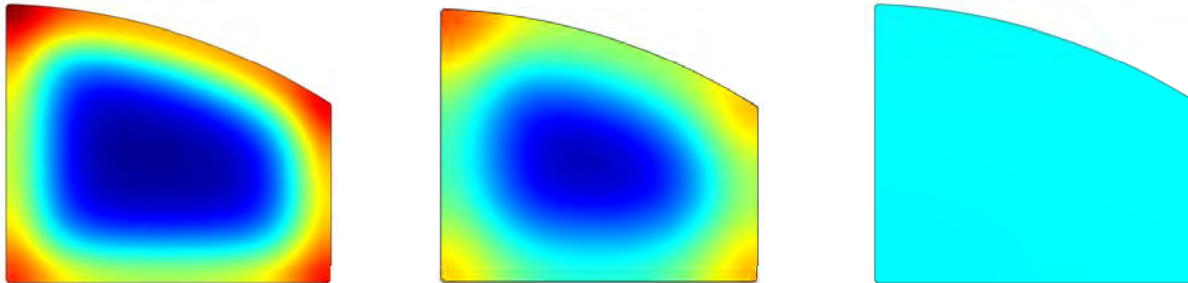


Formulation of problem as PDEs

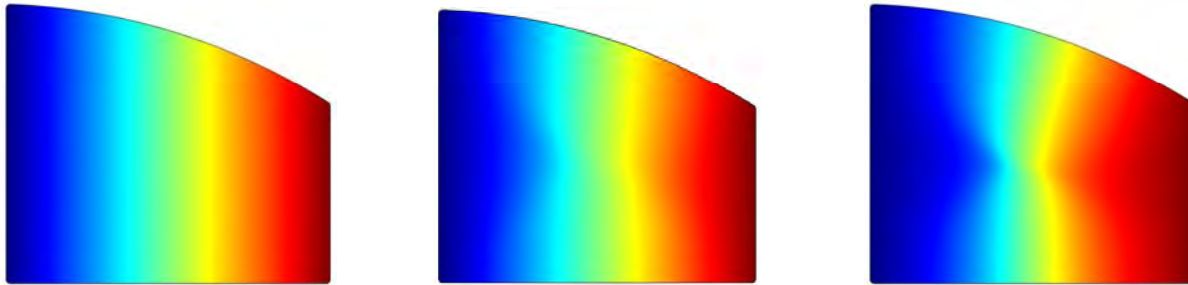
- PDE #1: diffusion, no convection, zero flux boundary conditions, initial value is the error estimate from the Laplacian temperature field from heat conduction problem
- PDE #2 & 3: convection of fields following Gastner-Newman method, no diffusion, zero flux boundary conditions
 - Velocity = - (gradient of error)/error
 - #2 initial value is the x coordinate
 - #3 initial value is the y coordinate
- As the error “diffuses”, the coordinate values are moved, as were the boundaries in the cartogram problem
- When the error is equi-distributed, the new coordinates can be used to transform the sensor locations

Illustration of “error diffusion”

Error redistribution



x coordinate redistribution



y coordinate redistribution

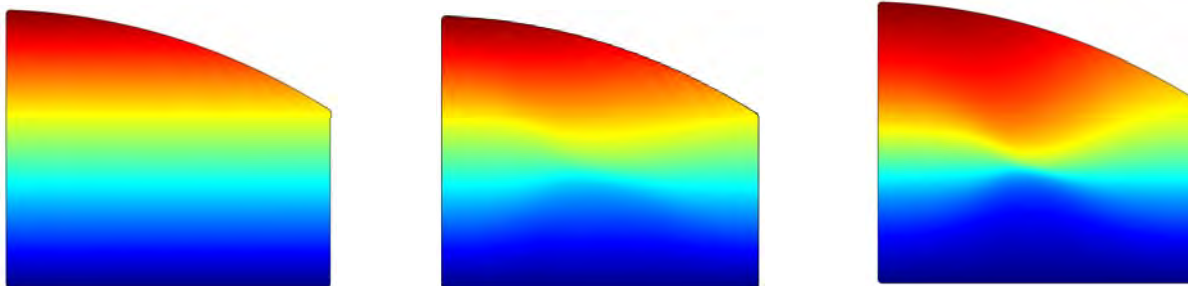
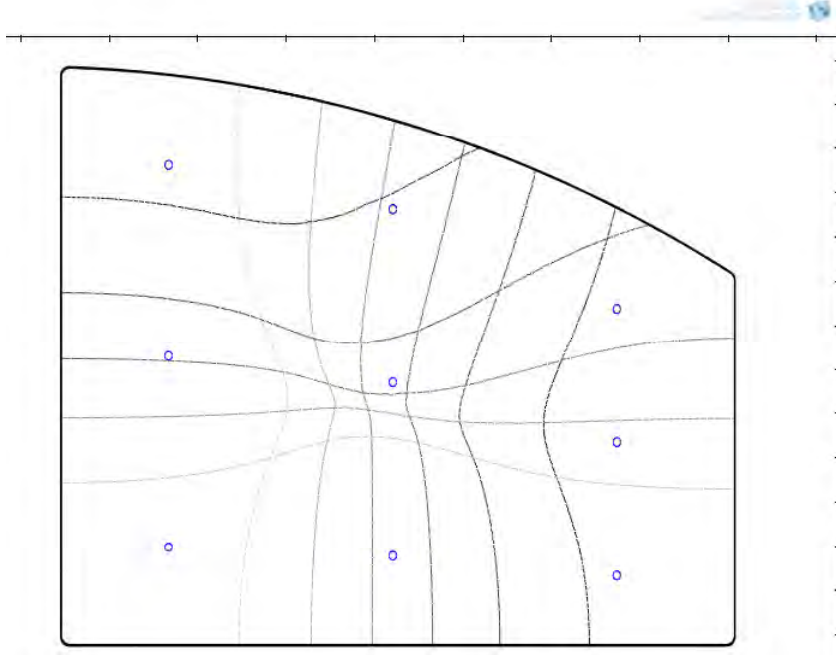
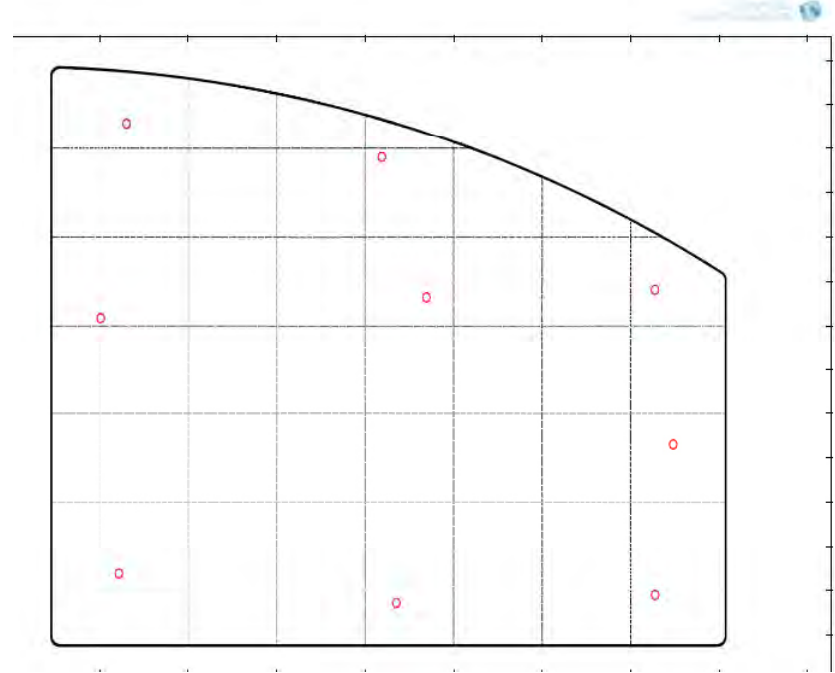


Illustration of Sensor Redistribution

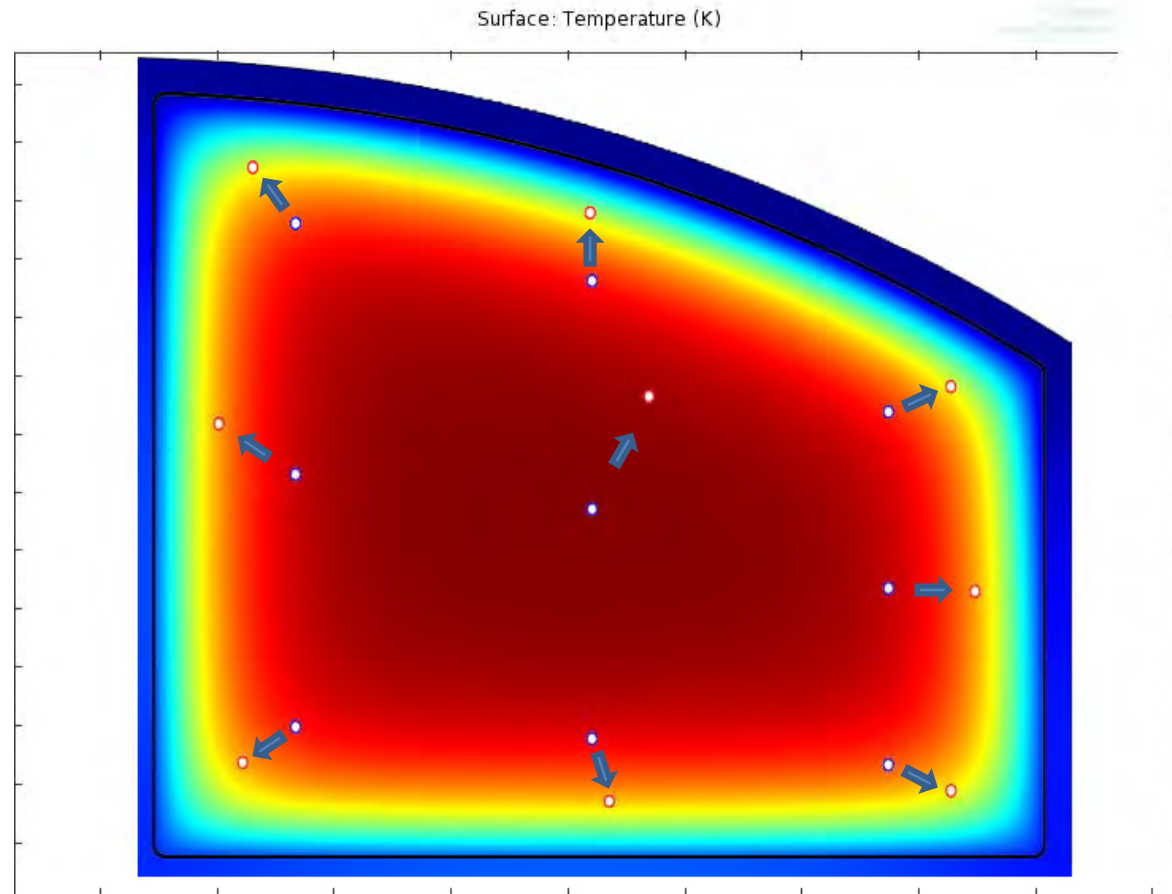
Uniform point distribution
on the equal-error coordinates



Optimal point distribution
on the original coordinates



Final Points Placement



Results:

- Constructing cartograms this way illustrates the Gastner-Newman approach as a diffusion model.
- The distribution of thermocouple points this way is a much more fundamental approach compared to statistical random sampling procedures.

Conclusion:

- While building cartograms may not be in the project description of many engineers, the general approach to locating sensors is of great utility in many situations.
- The present work is in a two dimensional space, and extension to three dimensions is straight forward.
- The present work is optimal for a single point in time, but there is a possibility that the method could be extended to optimize sensor placement over a range of times.