Energetics of Half-Quantum Vortices

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Outline

What is a quantum vortex?

- Magnetic fields of sufficient magnitude produce normal (N) regions in superconductors (SC)-the intermediate state.

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- In Type I superconductors the surface energy between SC and N regions is postive, and the intermediate state consists of macroscopic SC and N regions.

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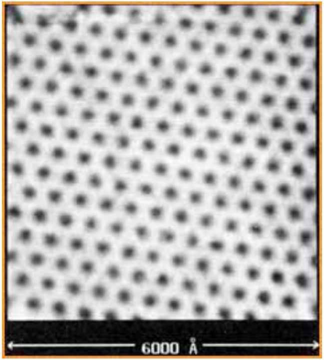


H. F. Hess et al., *Phys. Rev. Lett.*, **62**, 214 (1989)

- In Type II superconductors, the surface energy is negative, and the system seeks to increase the surface area.
- There is a quantum limit to how finely the regions can proliferate and how little flux is allowed penetrate the sample. These penetrating quanta of flux ($h/2e$) form a hexagonal Abrikosov lattice.

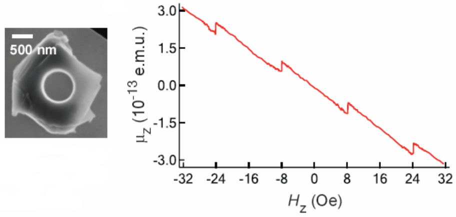
What is a half-quantum vortex? Why do I care?

- A vortex with a magnetic flux of h/4e, of course.
- Indicative of p-wave Cooper pairing. The electrons are bound in a spin-triplet state.
- HQVs have non-trivial topological properties enabling interesting quantum mechanics-possibly quantum computation.
- COMSOL provides us with a tool to solve the complicated non-linear equations derived from Ginzburg-Landau theory.

Outline

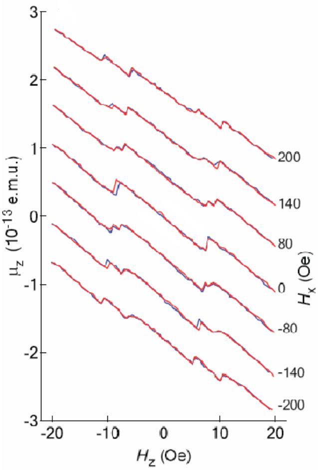
HQVs in SRO

- Properties of SRO suggest p-wave pairing. What about HQVs?
- A HQV's free energy diverges logarithmically with system size so one needs micron size samples to detect.
- Jang, Budakian, et al manufactured mesoscopic rings of SRO and measured the magnetic moment using magnetic cantilever techniques[1].



[1] J. Jang, et al, Observation of half-height magnetization steps in Sr_2RuO_4, *Science*, **331**, 6014 (2011)

Stability Wedge



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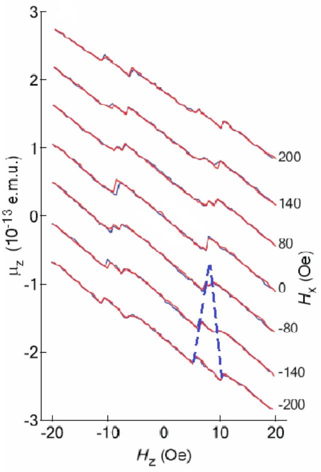

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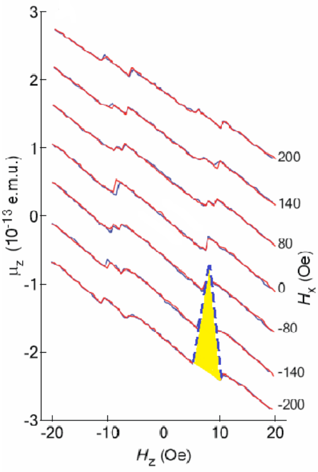

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$$F = \int d^3 r \left\{ \sum_{i=\uparrow,\downarrow} \left[-|\psi_i|^2 + \frac{1}{2}|\psi_i|^4 + \left| \left(\frac{\nabla}{\kappa} - i\mathbf{A} \right) \psi_i \right|^2 \right] \right.$$
$$\left. +2\beta \mathbf{J}_\uparrow \cdot \mathbf{J}_\downarrow + \boldsymbol{\mu}_{HI} \cdot \mathbf{B} + (\mathbf{B} - \mathbf{B}_{ext})^2 \right\}$$

where $\mathbf{J}_i = \mathrm{Re}\left\{ \psi_i^* \left(\frac{\nabla}{\kappa} - i\mathbf{A} \right) \psi_i / 2i \right\}$.

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- Half flux state exists when the total winding number $n_\uparrow + n_\downarrow$ is odd.

Kinematic Spin Polarization

- If we make the correspondences $|\psi|^2 = \rho$ and $\mathbf{v}_s = \nabla\theta/\kappa - \mathbf{A}$ the GL equations for a single order parameter are

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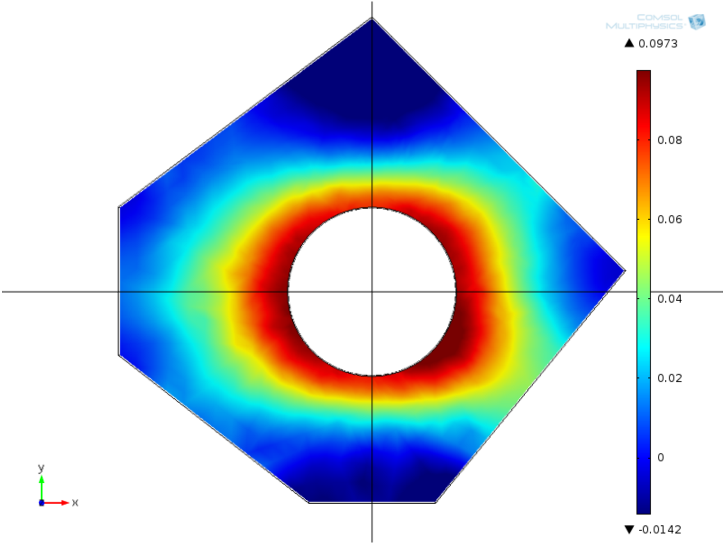
- Victor Vakaryuk and Tony Leggett showed that a difference in densities occurs in the HQV since the fluid velocities are different[2].
- A difference in densities is interpreted as a spin polarization lying in the ab-plane which can be used to lower the half flux state's free energy via the term

$$\boldsymbol{\mu}_{HI} \cdot \mathbf{B} = \mu \left(|\psi_\uparrow|^2 - |\psi_\downarrow|^2 \right) B_{ab}$$

[2] Vakaryuk, Leggett, *Phys. Rev. Lett.*, **103**, 057003 (2009)

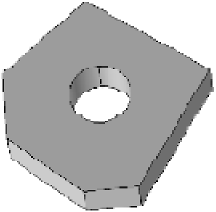
$\Delta|\psi|^2$ in the HQV

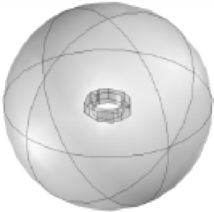


Outline

Simulation Geometry

- I drew a ring geometry which was similar to those that were used in the experiment.
- The ring was placed in a surrounding spherical volume.





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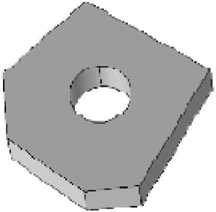

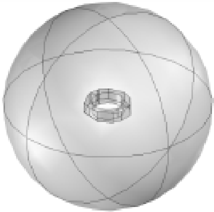

- Modeling the free space around the ring is important to properly calculate the increase in free energy due to the diamagnetism of the sample because one does not know a priori what the field is at the edge of the sample.

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$$\frac{\partial \psi_\uparrow}{\partial t} = -\frac{\delta F}{\delta \psi_\uparrow^*}, \quad \frac{\partial \psi_\downarrow}{\partial t} = -\frac{\delta F}{\delta \psi_\downarrow^*}, \quad \frac{\partial \mathbf{A}}{\partial t} = -\frac{\delta F}{\delta \mathbf{A}}$$

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- The resulting equations are daunting.

Equations

Do you want to solve these analytically? Here are the equations

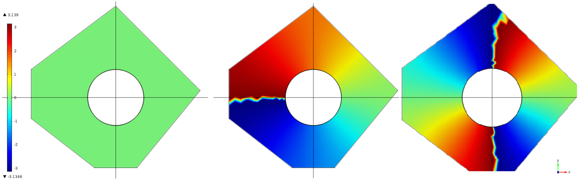
- $\frac{\partial \psi_i}{\partial t} - \nabla \cdot \left(\frac{1}{\kappa^2} \nabla \psi_i - \frac{i}{\kappa} \psi_i \mathbf{A} - \frac{\beta}{i\kappa} \psi_i \mathbf{J}_j \right) =$
  $\left(|\psi_i|^2 - A^2 + 2\beta \mathbf{J}_j \cdot \mathbf{A} \right) \psi_i - \left(\frac{i}{\kappa} \mathbf{A} + \frac{\beta}{i\kappa} \mathbf{J}_j \right) \cdot \nabla \psi_j$
- $\frac{\partial \mathbf{A}}{\partial t} - \nabla^2 \mathbf{A} = \mathbf{J}_\uparrow + \mathbf{J}_\downarrow + \beta \left(|\psi_\uparrow|^2 \mathbf{J}_\downarrow + |\psi_\downarrow|^2 \mathbf{J}_\uparrow \right)$
- $\mathbf{J}_i = \text{Re} \left\{ \psi_i^* \left(\frac{\nabla}{\kappa} - i\mathbf{A} \right) \psi_i / 2i \right\}$

and boundary conditions

- $\hat{\mathbf{n}} \cdot \left(\frac{1}{\kappa^2} \nabla \psi_i - \frac{i}{\kappa} \psi_i \mathbf{A} - \frac{\beta}{i\kappa} \psi_i \mathbf{J}_j \right) = 0$
- $\hat{\mathbf{n}} \times (\mathbf{B} - \mathbf{B}_{ext}) = 0$

Initial Conditions

- The vector potential was set on the edge of the sphere, fixing the amount of flux through the system.
- The initial conditions for the OPs were set to fix the winding numbers.



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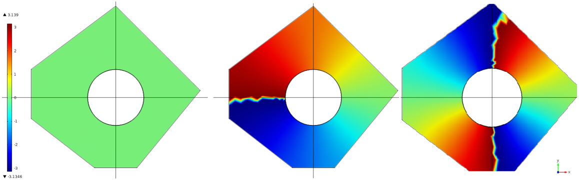


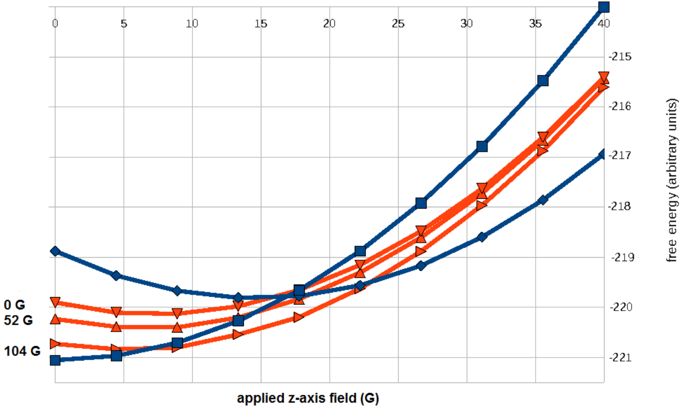
- The winding number/flux state of the OP is topolgically protected. COMSOL will find an equilibrium state **given that winding number**, even if another winding number is more stable.

$\rightarrow$ Calculate free energies for all states and compare.

Outline

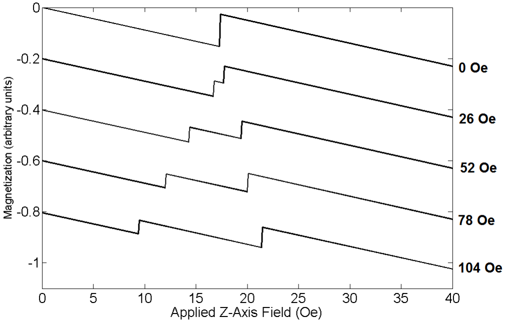
Free Energy Curves

- After solving the GL equations for the particular flux state and magnetic field, the free energy could be calculated by performing the appropriate integral.
- Produce free energy diagrams of integer and half flux states.



Magnetization Curves

- One can then calculate $-\frac{\partial F}{\partial B_{z,ext}}$ to produce magnetization curves.



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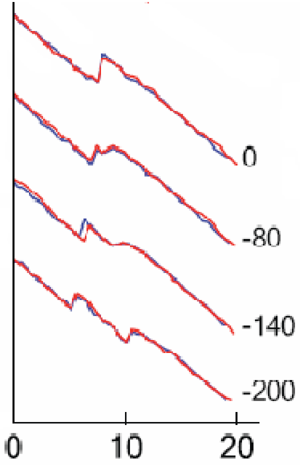

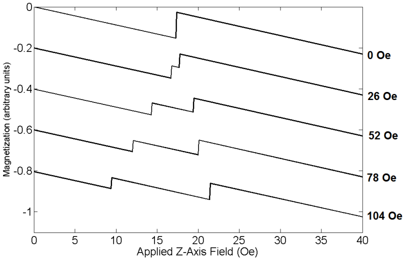

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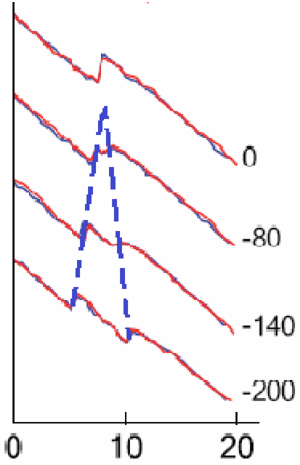

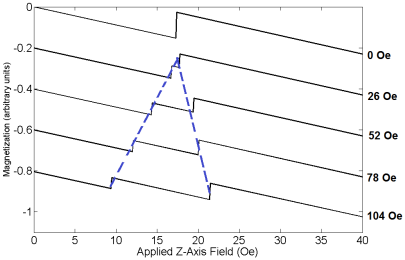

Outline

- COMSOL shows that the GL model can qualitatively and quantitativly reproduce the experimental data.
- Drawing a general phase diagram in (β, μ) space isn't possible. The energetics depends on the geometry so there is a phase diagram for every ring.
- Acquiring accurate values of β and μ are difficult since we don't know exactly what portions of the ring are superconducting. Unconventional superconductivity is very sensitive to crystal defects and damage during construction is inevitable.