

Einstein vs Newton: Geodesic Free-Fall Trajectories Approaching Massive Black Hole

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Introduction: The free-fall trajectories of objects near a massive black hole are treated using Einstein's “*General Theory of Relativity*” (GR) and are compared to corresponding solutions using “*Newton's Universal Law of Gravitation*” (NULG). The effect of Einstein's $\mathbb{S}=1/(r-R_s)$ ODE singularity coefficient on trajectories approaching the black hole (as $r \rightarrow R_s$) is investigated where $R_s \equiv$ Schwarzschild radius $2GM/c$.

Assistance with numerical integration as $\mathbb{S}$ goes from $+\infty$ to $-\infty$ as r crosses $r=R_s$ is replaced with a smooth cutoff transition function $\mathbb{S}(r)_n$ (Fig.3 inset with four 1.2×10^n $\{n=4,2,-2,-6\}$ peak cutoff off values).

Schwarzschild coordinates are used in GR space-space metric $ds^2 = -c^2 dt^2(r-R_s)/r + r^2 S dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

Einstein's **GR** Geodesic Equations

$$\left[\begin{aligned} \frac{d^2 t}{d\lambda^2} + \mathbb{S} \frac{R_s}{r} \frac{dr}{d\lambda} \frac{dt}{d\lambda} &= 0 \quad \left(\mathbb{S} \equiv \frac{1}{r-R_s} \right) \\ \frac{d^2 r}{d\lambda^2} + \frac{GM}{r^3} (r-R_s) \left(\frac{dt}{d\lambda} \right)^2 - \mathbb{S} \frac{R_s}{2r} \left(\frac{dr}{d\lambda} \right)^2 - \\ (r-R_s) \left[\left(\frac{d\theta}{d\lambda} \right)^2 + \sin^2(\theta) \left(\frac{d\phi}{d\lambda} \right)^2 \right] &= 0 \\ \frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} - \frac{\sin(2\theta)}{2} \left(\frac{d\phi}{d\lambda} \right)^2 &= 0 \\ \frac{d^2 \phi}{d\lambda^2} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{dr}{d\lambda} + 2 \cot(\theta) \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} &= 0 \end{aligned} \right]$$

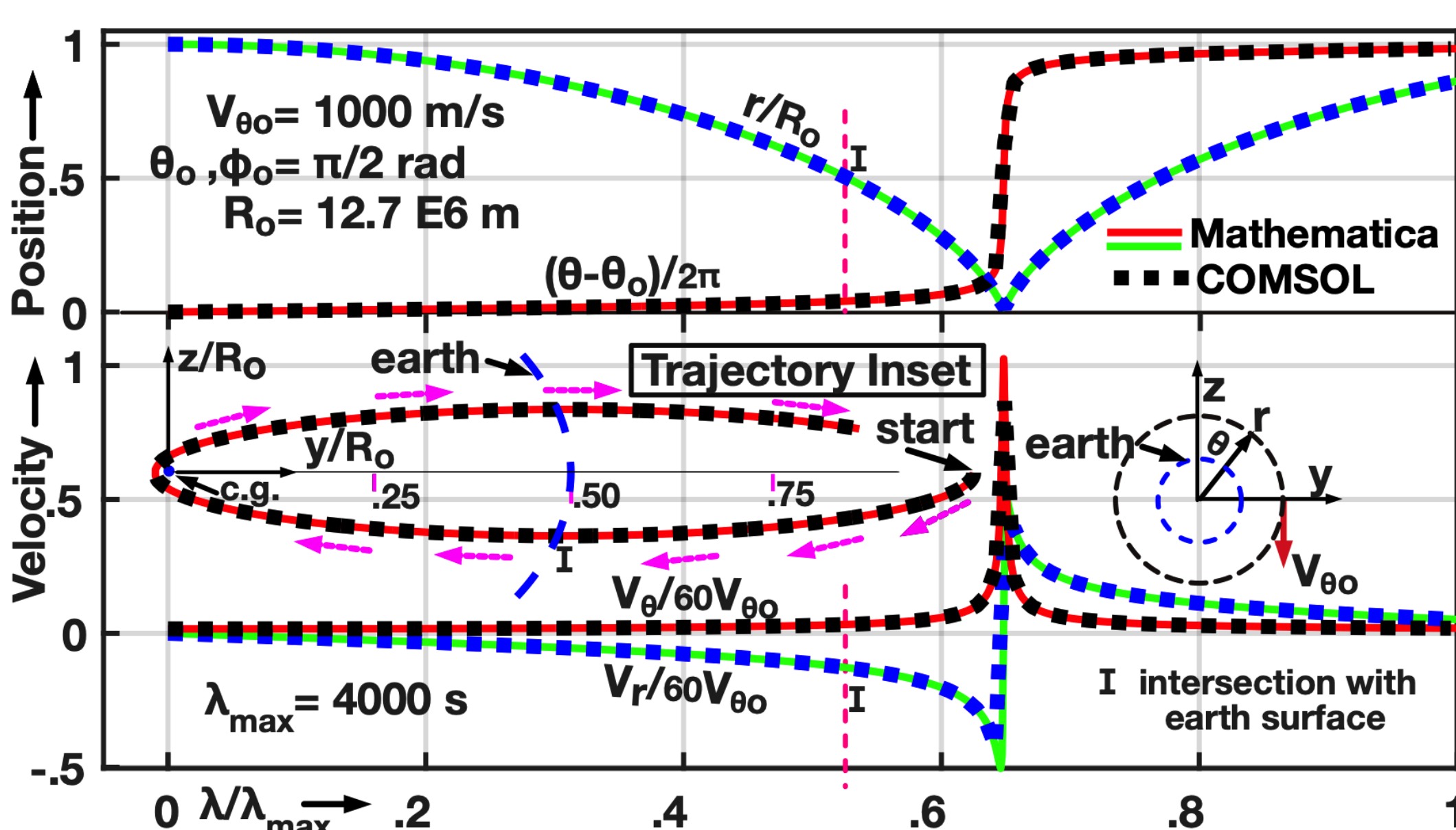
Newton's **NULG** Geodesic Equations

$$\left[\begin{aligned} \frac{d^2 r}{dt^2} + \frac{GM}{r^2} - r \left[\left(\frac{d\theta}{dt} \right)^2 + \sin^2(\theta) \left(\frac{d\phi}{dt} \right)^2 \right] &= 0 \\ \frac{d^2 \theta}{dt^2} + \frac{2}{r} \frac{d\theta}{dt} \frac{dr}{dt} - \frac{\sin(2\theta)}{2} \left(\frac{d\phi}{dt} \right)^2 &= 0 \\ \frac{d^2 \phi}{dt^2} + \frac{2}{r} \frac{d\phi}{dt} \frac{dr}{dt} + 2 \cot(\theta) \frac{d\theta}{dt} \frac{d\phi}{dt} &= 0 \end{aligned} \right]$$

Computational Method: COMSOL uses the *Mathematics Module* (& cross checking Mathematica Code uses the *NDSolveValue* option) to solve above equations [1] for the trajectories and corresponding position & velocity time histories.

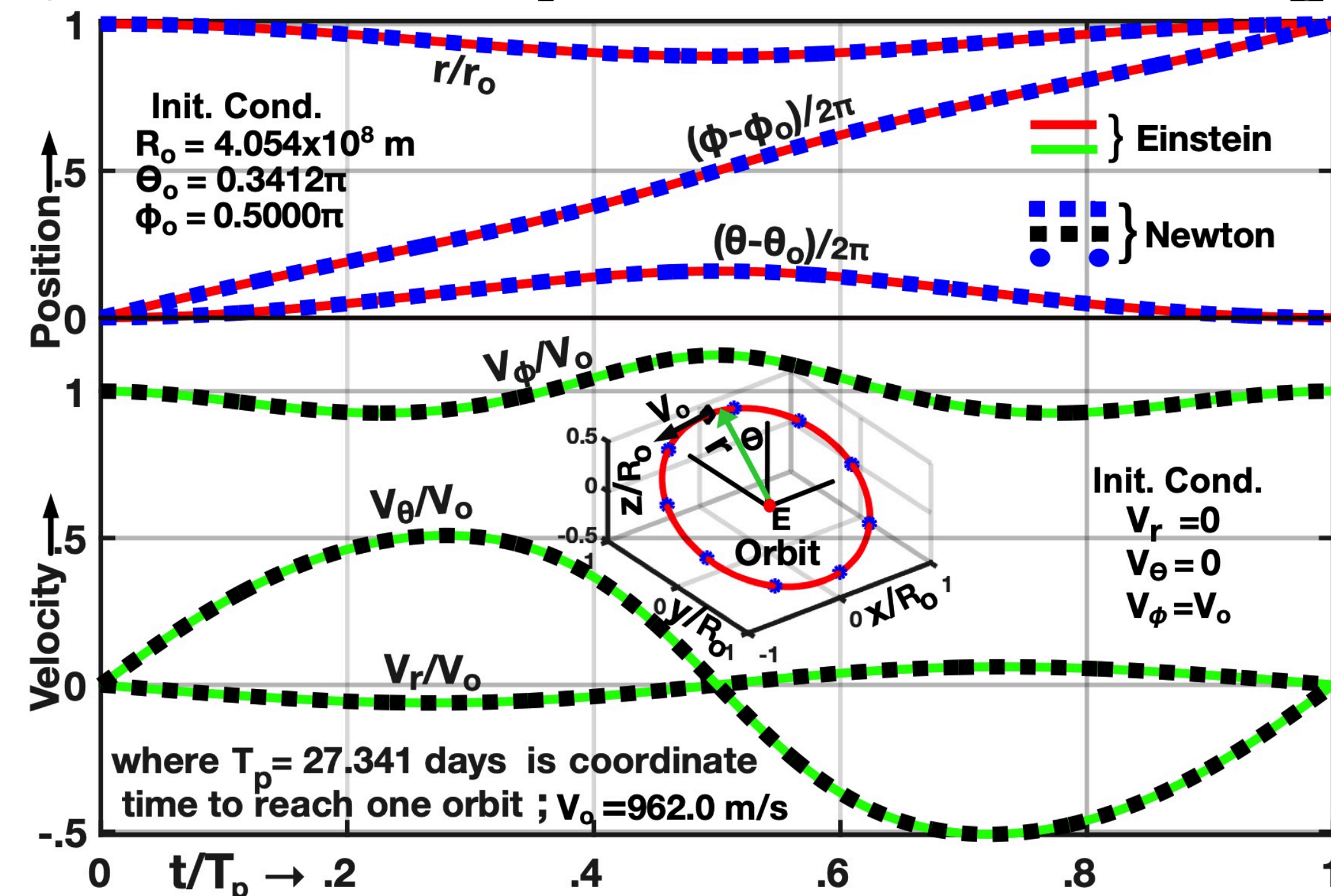
Approximations of the GR Christoffel coefficients $\Gamma_{tr}^t = GMS/r \approx GMS_n/r$ & $\Gamma_{rr}^r = -\Gamma_{tr}^t \approx -GMS_n/r$ are used which amounts to replacing $\mathbb{S} \rightarrow \mathbb{S}(r)_n$ in above GR equations.

Results: • Fig.1 Satellite Touchdown: For starting altitude $r=R_0=12.7E6m$ (with $R_s=0.0088m$), a satellite falls towards Earth's surface and touches down as indicated by marker I. A COMSOL- Mathematica motion validation vs. λ/λ_{max} proper time & x-z plane trajectories are shown. For fun we integrate past touchdown, as if the earth's mass was a concentrated point.



• Fig.2 Moon Orbits Earth: Wikipedia moon's apogee init. cond. data: for pos.; $\theta_0=90^\circ-23.44^\circ$ (earth obliquity) -5.14° (lunar orbital incline) $= 61.42^\circ$ & $\phi_0=90^\circ$ with vel. $v_{\phi 0}=962$ m/s. The starting radius $R_0=4.05E8m \gg R_s=0.00886488m$, therefore

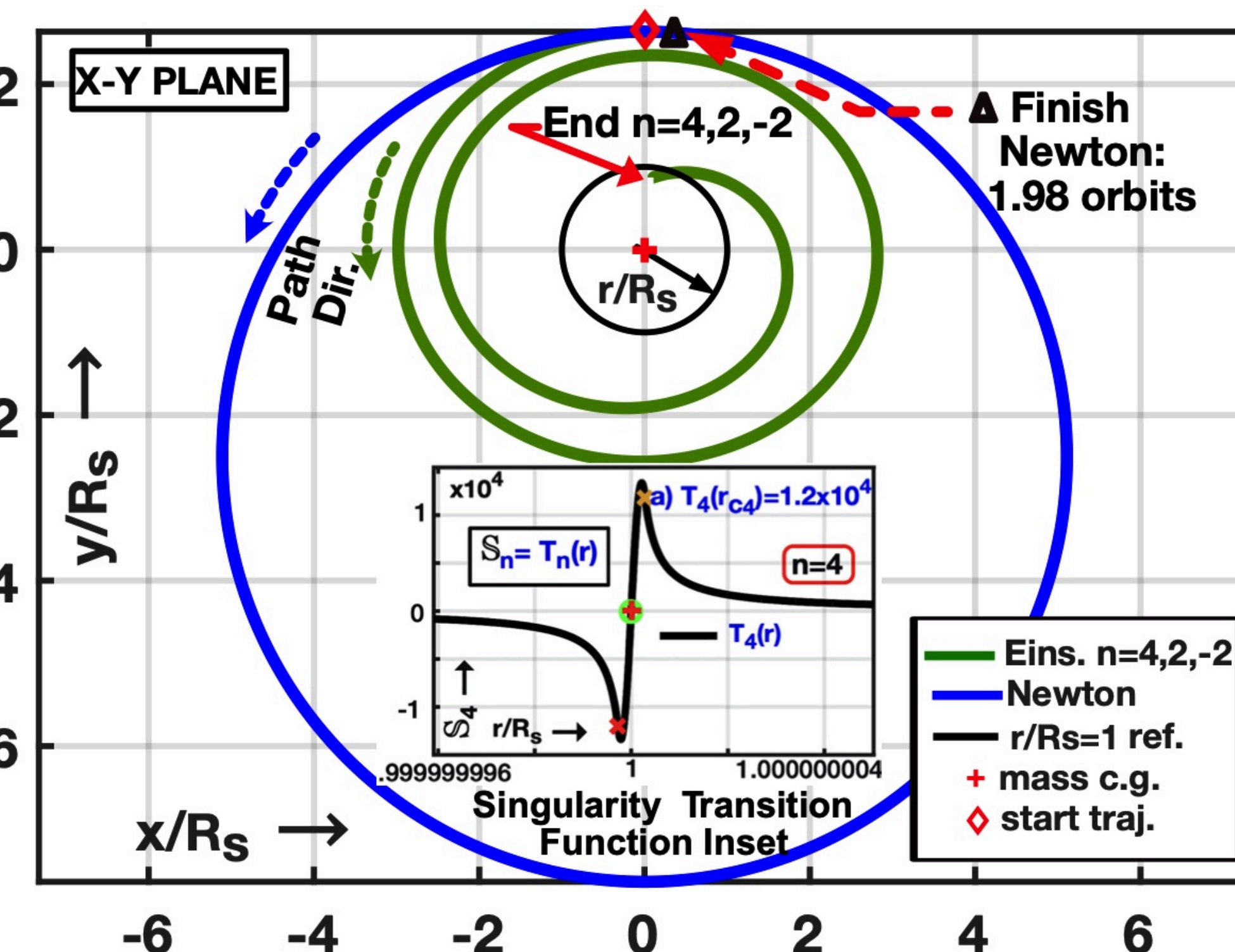
the $\mathbb{S}$ term $\rightarrow \pm\infty$ never comes into play, thus GR and NULG are very close. Both theories give the same average



vel. $\int |v| d\lambda / T_p = 1019$ m/s; r_a/r_p apogee/perigee ratio $= 1.122$; and period $T_p = 27.343$ days vs. Wikipedia's $\{1022$ m/s, 1.1159 , 27.332 days $\}$.

• Fig.3 Near Star Death Spiral y vs. x Trajectories:

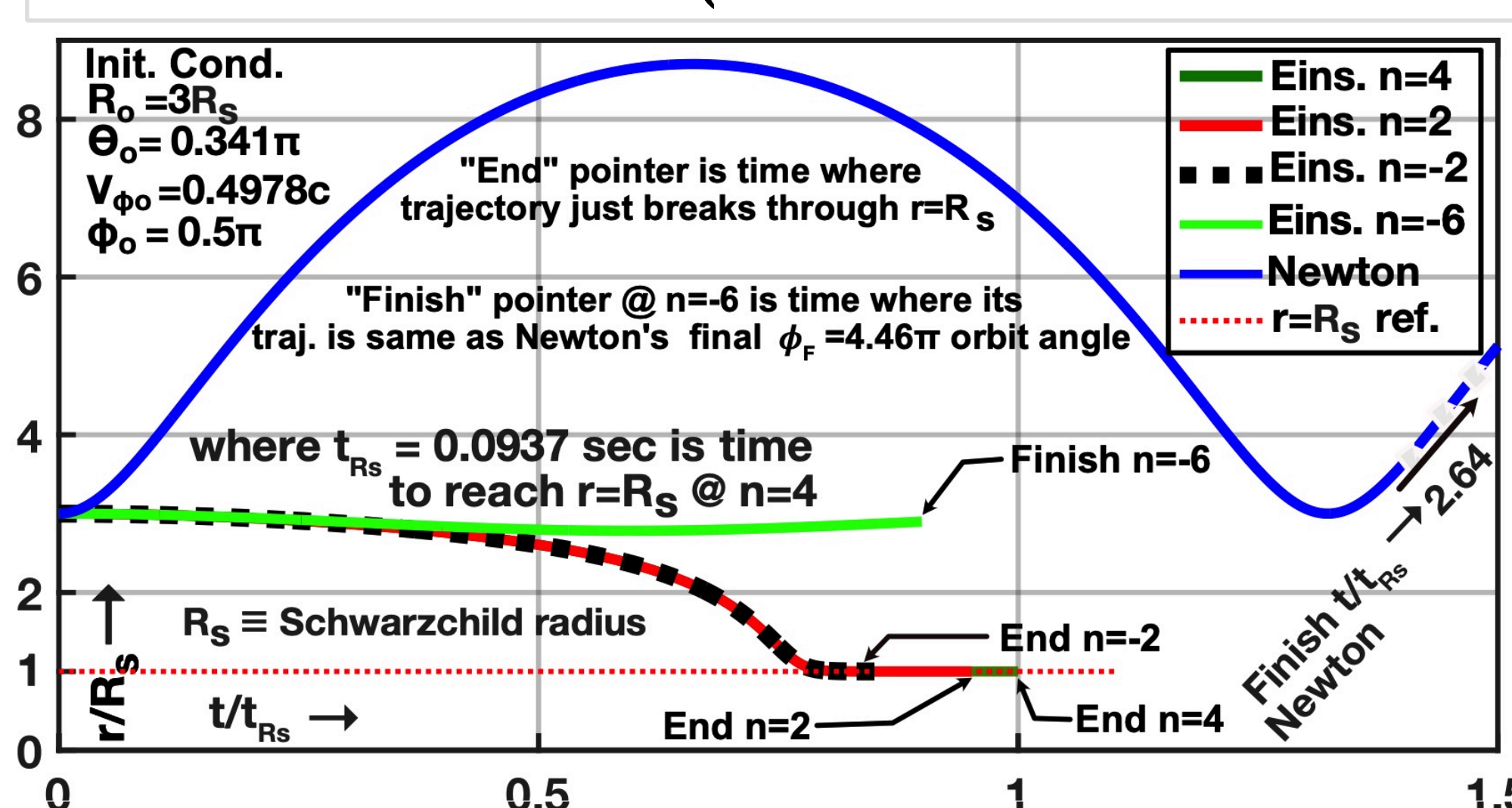
The primary mass is the size of the KY Cygni star whose mass is 100 times the mass of our sun. The star is already assumed to be collapsed into a point mass



and motions experienced by a secondary body approaching with a large tangential velocity is investigated as $r \rightarrow R_s$. The same starting angles $\{\phi_0 \theta_0\}$ of Fig.2 are arbitrarily used.

• Fig.4 Near Star Death Spiral r/Rs vs. Time:

When using Schwarzschild coordinates, the r position vs. *coordinate time* t/t_{Rs} curve shows a flattening of the response, while never passing the point $r=R_s$ (called coordinate freeze [1]). However using the $\mathbb{S}_n$ function allows the numerical solution to break thru $r=R_s$ at time t_b (labeled “End n=” in plot) depending on



the size of the singularity strength parameter n . Bigger n gives bigger t_b values. There are substantial differences in Figs.(3-4) GR vs NULG theories.

Conclusions: Examples are solved for the GR & NULG time histories where trajectories in the Earth's gravitational field showed little difference between theories, however the geodesic paths thru star size gravitational fields are quite different.

References: 1. S. Carroll. *General Space-time and Geometry*, Cambridge University Press, (UK 2017).