

Einstein vs Newton: Geodesic Free-Fall Trajectories Approaching Massive Black Hole

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Abstract: The free-fall trajectories of objects (geodesics) near a massive black hole are treated using Einstein's "*General Theory of Relativity*" equations and are compared to corresponding solutions using "*Newton's Universal Law of Gravitation*" equations. The effect of Einstein's $\mathbb{S}=1/(r-R_s)$ ODE singularity coefficient on trajectories approaching the black hole (as spherical coordinate $r \rightarrow R_s$) is investigated, where R_s is the Schwarzschild radius $2GM/c^2$. Assistance with numerical integration (as $\mathbb{S}$ goes from $+\infty$ to $-\infty$ when r crosses $r=R_s$) is treated with a Fig.(1) smooth cutoff transition function $\mathbb{S}_n(r)$ with peak cutoff off values 1.2×10^n { $n=4,2,-2,-6$ }.

Keywords: Geodesics, Time Dependent Nonlinear ODE's, General Relativity

1. Introduction

The paper illustrates the use of COMSOL for obtaining the position vector solution $\vec{x}(\lambda)$ as a solution to Einstein's *time dependent General Relativity Theory nonlinear ODE's* equations (GR) Ref.[1], where λ is a proper time parameter for advancing the position. For comparative purposes, it is of interest to examine how GR compares to the same solution using *Newton's Universal Law of Gravitation (NULG)* equations. Four example illustrations are addressed: 1) satellite crashes into earth 2) Moon orbits earth, 3) plunge fall towards massive black hole starting at $r=R_0=1.5R_s$ and 4) death spiral fall towards black hole starting at larger $r=R_0=3R_s$. In situations where we are far from the $r \rightarrow R_s$ region, we shall see that GR and NULG essentially overlay each other such as in example 2. However in examples 3 and 4 where we are in the region of $r \rightarrow R_s$, substantial differences between the two theories are shown.

2. Governing Equations/Theory

2.1 Einstein's GR Geodesic Equations

The rewritten Ref.[1] GR Geodesic ODEs in Schwarzschild spherical coordinates r, θ, ϕ (Fig.(10)

inset) with independent variable λ representing proper time, and explicitly putting back the speed of light c (physicist like using units where c is =1) is given by:

$$\frac{d^2 t}{d\lambda^2} + \mathbb{S} \frac{R_s}{r} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0 \quad \left(\mathbb{S} \equiv \frac{1}{r-R_s} \quad R_s \equiv \frac{2GM}{c^2} \right) \quad (1)$$

$$\frac{d^2 r}{d\lambda^2} + \frac{GM}{r^3} (r-R_s) \left(\frac{dt}{d\lambda} \right)^2 - \mathbb{S} \frac{R_s}{2r} \left(\frac{dr}{d\lambda} \right)^2 - (r-R_s) \left[\left(\frac{d\theta}{d\lambda} \right)^2 + \sin^2(\theta) \left(\frac{d\phi}{d\lambda} \right)^2 \right] = 0 \quad (2)$$

$$\frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} - \frac{\sin(2\theta)}{2} \left(\frac{d\phi}{d\lambda} \right)^2 = 0 \quad (3)$$

$$\frac{d^2 \phi}{d\lambda^2} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{dr}{d\lambda} + 2 \cot(\theta) \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} = 0 \quad (4)$$

The singularity coef. is defined as $\mathbb{S}$ and involves the Schwarzschild radius R_s (black hole event horizon).

2.2 Newton's NULG Geodesic Equations

These can be directly obtained from Newton's NULG force of attraction law written in spherical coordinates -or- alternatively reduced from Eqs.(2-4) by letting $c \rightarrow \infty$ ($\therefore R_s \rightarrow 0$), $\lambda \rightarrow t$ and $dt/d\lambda \rightarrow 1$.

$$\frac{d^2 t}{dt^2} + \frac{GM}{r^2} - r \left[\left(\frac{d\theta}{dt} \right)^2 + \sin^2(\theta) \left(\frac{d\phi}{dt} \right)^2 \right] = 0 \quad (5)$$

$$\frac{d^2 r}{dt^2} + \frac{2}{r} \frac{dr}{dt} \frac{d\theta}{dt} - \frac{\sin(2\theta)}{2} \left(\frac{d\phi}{dt} \right)^2 = 0 \quad (6)$$

$$\frac{d^2 \phi}{dt^2} + \frac{2}{r} \frac{d\phi}{dt} \frac{dr}{dt} + 2 \cot(\theta) \frac{d\theta}{dt} \frac{d\phi}{dt} = 0 \quad (7)$$

2.3 GR v_r, v_θ, v_ϕ Physical Velocity Components

A relation between the primitive variable rates $\{dr/d\lambda, d\theta/d\lambda, d\phi/d\lambda\}$ and the *physical velocity* components $\{v_r, v_\theta, v_\phi\}$ is needed for both setting initial conditions and plotting results and is obtained from the Eq.(8) *space like* Schwarzschild metric Ref.[1] and related Ref.[2] Eq.(9) *time like* metric (written with Eqs.(10) $g_{tt}, g_{\mu\mu}$ metric coefficients):

$$ds^2 = g_{tt} dt^2 + \sum_{\mu=r,\theta,\phi} g_{\mu\mu} d\mu^2 = g_{tt} dt^2 + \sum_{\mu=r,\theta,\phi} dl_\mu^2 \quad (8)$$

$$d\lambda^2 = -g_{tt} dt^2 / c^2 - (1/c^2) \sum_{\mu=r,\theta,\phi} g_{\mu\mu} d\mu^2 \quad (9)$$

$$g_{rr} = (1 - R_s/r)^{-1}; \quad g_{tt} = -c^2 / g_{rr}; \quad g_{\theta\theta} = r^2; \quad g_{\phi\phi} = (r \sin \theta)^2 \quad (10)$$

$$x = r \sin(\theta) \cos(\phi); \quad y = r \sin(\theta) \sin(\phi); \quad z = r \cos(\theta) \quad (11)$$

The Eq.(12b) *physical velocity* Ref.[4] v_μ in the μ^{th} direction is defined as the ratio of the proper length $d\mu = d\mu \sqrt{g_{\mu\mu}}$ (i.e. $d\mu$ from 2nd Eq.(8)) divided by the Eq.(9) proper time $d\lambda_{\text{st}} = (dt/c)\sqrt{-g_{tt}}$ (for a local *static observer* at constant $\{r, \theta, \phi\}$), so:

$$\text{for a fixed } \mu, \quad d\mu = 0 \quad \therefore \quad d\lambda_{\text{st}} = \sqrt{-g_{tt}} \, dt/c \quad \mu = r, \theta, \phi \quad (12a)$$

$$v_\mu = d\mu/d\lambda_{\text{st}} = c \frac{d\mu}{dt} \sqrt{g_{\mu\mu}} / -g_{tt} = c \frac{d\mu/d\lambda}{dt/d\lambda} \sqrt{g_{\mu\mu}} / -g_{tt} \quad (12b)$$

The Eq.(12b) v_μ is known by other names (e.g. *velocity measured by static observer*; *velocity measured by shell observer*; and the Ref.[4,6] *physical velocity*) where herein we adopt the latter name *physical velocity*.

2.4 GR v'_r, v'_θ, v'_ϕ Coordinate Velocity Components

The coordinate velocities (denoted here with primes) are rates of change w.r.t. coordinate time t . The contrast to *physical velocity* (with Eqs.(10) $g_{tt}, g_{\mu\mu}$ coeffs. plugged into Eq.(12b)) is illustrated for the r^{th} velocity comp., where the $(1-R_s/r)^{-1}$ multiplier relates the two.

$$v'_r = \frac{dr}{dt} = \frac{dr}{d\lambda} / \frac{dt}{d\lambda} \quad \text{vs.} \quad v_r = (1 - R_s/r)^{-1} \frac{dr}{d\lambda} / \frac{dt}{d\lambda} \quad (13)$$

Coord. Vel. Coord. Vel. v'_r

2.5 Initial Integration Conditions for GR

The 1st 4 of 8 initial conditions on $\{t, r, \theta, \phi\}$ are straight forward and are the starting time and location of the falling body at $\lambda = \lambda_o$. The GR initial conditions have an unusual situation compared to conventional Newtonian mechanics, where proper time λ is the independent variable with coordinate time $t(\lambda)$ & $dt(\lambda)/d\lambda$ as dependent variables, and therefore require initial integration constants t_o and $(dt/d\lambda)_o$ at $\lambda = \lambda_o$. The remaining 4 rates of change initial conditions $\{(dt(\lambda)/d\lambda)_o, (d\mu(\lambda)/d\lambda)_o \mid \mu = r, \theta, \phi\}$ are more involved and are given by:

$$8 \text{ Initial cond. @ } \lambda = \lambda_o \text{ with start velocities } v_{r_o} \, v_{\theta_o} \, v_{\phi_o} \\ t = t_o ; \, r = R_o ; \, \theta = \theta_o ; \, \phi = \phi_o \quad (14)$$

$$(dt/d\lambda)_o = 1/\sqrt{(1-R_s/R_o)(1-\beta_o^2)} ; \beta_o^2 = \sum_{\mu=r,\theta,\phi} (v_{\mu_o}/c)^2 \quad (15)$$

$$(d\mu/d\lambda)_o = v_{\mu_o} (1/c) (dt/d\lambda)_o \sqrt{-g_{tt_o}/g_{\mu\mu_o}} \quad \mu = r, \theta, \phi \quad (16)$$

Equation (16) is obtained by evaluating Eq.(12b) at $\lambda = \lambda_o$ and algebraically solving for $d\mu(\lambda)/d\lambda_o$ (e.g. for $\mu=r$, filling in the Eq.(10) metrics, gives the relation $(dr(\lambda)/d\lambda)_o = v_{r_o}(1-R_s/R_o)(dt/d\lambda)_o$). Equation(15) is obtained by first multiplying Eq.(9) by $c^2/d\lambda^2$, inserting Eq(16) for $d\mu(\lambda)/d\lambda_o$, then inserting metric coefficients Eqs.(10) and finally algebraically solving for $(dt/d\lambda)_o$. Note that the probative purely radial input case $v_r = v_{r_o} ; v_{\theta_o} = 0 ; v_{\phi_o} = 0$, and $R_o \gg R_s$, reduces to the Ref.[3] special relativity rate $(dt/d\lambda)_o = 1/\sqrt{(1-\beta^2)}$, where $\beta^2 = (v_{r_o}/c)^2$.

2.6 Initial Integration Conditions for NULG

The six NUGL initial conditions on position are the $\{r, \theta, \phi\}$ falling body CG position at starting value $t = t_o = 0$. Given the $\{v_{r_o}, v_{\theta_o}, v_{\phi_o}\}$ initial velocities at t_o , the initial time rates, are obtained upon substituting these 3 into $\{(dr/dt)_o = v_{r_o}, (d\phi/dt)_o = v_{\phi_o}/R_o \sin(\theta_o), (d\theta/dt)_o = v_{\theta_o}/R_o\}$ (note these 3 can also be reduced from Eqs.(16), upon letting $c \rightarrow \infty$, and $d\lambda \rightarrow dt$).

2.7 Singularity Coefficient Approximation in GR

The $\mathbb{S}$ term (related to the g_{rr} metric via $\mathbb{S} = g_{rr}/r$) appearing in Eqs.(1-2), switches from $+\infty$ to $-\infty$ as r crosses the point $r=R_s$ (going right to left). Thus ODE numerical solutions have numerical difficulty in this

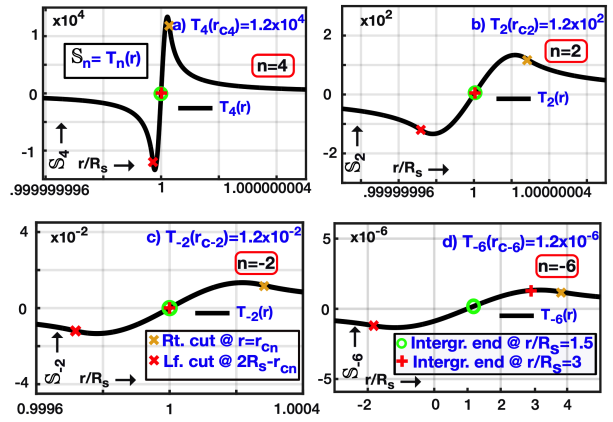


Figure 1 Transition $\mathbb{S}_n$ vs r/R_s for T_n cutoff peak 1.2×10^n

zone. We introduce an approximate $\mathbb{S}$ term with a transition function $\mathbb{S}_n = T_n(r)$ which replaces the $\pm \infty$ switching zone with a segment of a sine curve that smoothly blends into the original Eq.(1) $\mathbb{S}$ (e.g. Fig.(1a)).

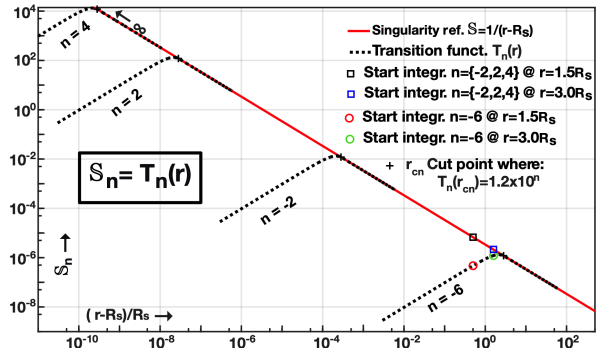


Figure 2 All Four $\mathbb{S}_n$ vs r/R_s curves on Same log-log Plot

The transition function is built by picking two anti-symmetrical cut off points $\{r_c, T_n(r_c)\}$ & $\{r'_c, T_n(r'_c)\}$ (where $T_n(r'_c) = -T_n(r_c)$ & $r'_c = 2R_s - r_c$) straddling the $\pm \infty$ singularity as shown by the $\times$ markers in Fig.(1).

Outside the cutoff zone, $T_n(r)=1/(r-R_s)$ for $r \geq r_c$ and $r \leq r'_c$ and inside by $T_n(r)=A_n \sin(k_n(r-R_s))$ for $r'_c \leq r \leq r_c$. For the smooth continuity of the 2 neighboring functions, equating $T_n(r)$ and slope $dT_n(r)/dr$ at the cutoff point yields 2 simultaneous eqs. for A_n & k_n . Solving gives: $A_n = T_n(r_c)/\sin(z^*)$, $k_n = z^* T_n(r_c)$ with $z^* = 2.02875$ as first >0 root of transcendental equation $z + \tan(z) = 0$, (where $z = k_n(r_c - R_s)$). Later we consider the effect of the Fig.(1) transition function on the ODE solutions for cutoff values $T_n(r_c) = 1.2 \times 10^n$ for $n = \{4, 2, -2, -6\}$. Log-Log graphs (for our large domain and range span) are used to plot all 4 curves together as shown in Fig.(2). Only the right side $r > R_s$ of the transition zone is plotted, where the straight red line corresponds to the actual un-approximated $S = 1/(r - R_s)$ function.

3. Method

COMSOL (& alternative software) is used to solve Eqs.(1-4) & Eqs.(5-7) for the trajectory, and the position & velocity time histories. COMSOL uses the Mathematics Module and Mathematica™ is used as a verification comparison (using *NDSolveValue*). Approximations of the GR Christoffel coeffs. $\Gamma_{tr}^t = GMS/r \approx GMS_n/r$ & $\Gamma_{rr}^r \approx -GMS_n/r$ are used which amounts to replacing $S \rightarrow S_n$ in Eqs.(1-2).

Values $c = 2.99729E8$ m/s & $G = 6.67E-11$ m³/s²kg for Exs. 1-4; $M = 5.97E24$ kg in Exs. 1-2; and $M = 1.989E32$ kg in Exs. 3-4 are used in the computations.

3.1 Ex. 1 COMSOL Satellite Touchdown Validation

A satellite with init. cond. $v_{\theta 0} = 1000$ m/s at $\theta_0 = \phi_0 = \pi/2$ & $(dt/d\lambda)_0 = 1.000000000353$ from Eq.(15), falls towards Earth and touches down at Earth radius $r = R_E = 6.37E6$ m as indicated by plot marker I. From the 3^d Eq.(1), $R_s = 0.00886488$ m, and the starting altitude is $r = R_0 = 12.7E6$ m. A good COMSOL-Mathematica motion comparison vs. $\lambda/\lambda_{\max}$ & the x-z plane trajectory are in Fig.(3). For fun we integrate past touchdown, as if the earth's mass was a concentrated point mass to observe the virtual flat orbit.

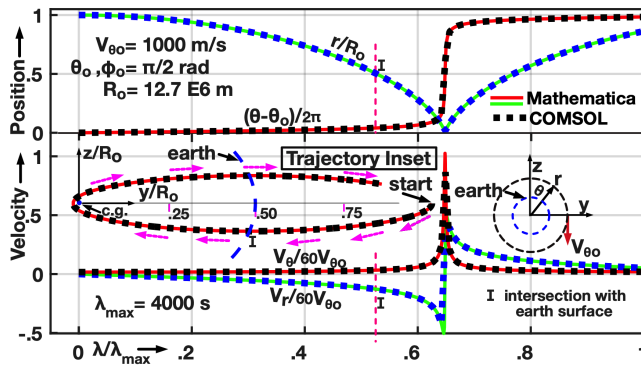


Figure 3 COMSOL Verification: Satellite Touchdown

4. Einstein's GR & Newton's NULG Comparisons

In the next 3 examples, Einstein's GR theory is compared to Newton's NULG theory and results are plotted vs. coordinate time t (rather than proper time λ) in order to have a consistent coordinate time comparison.

4.1 Ex. 2 Einstein vs Newton Moon Orbit

The Fig.(4) Wikipedia listed moon's init. cond. for velocity and position corresponds to observational data

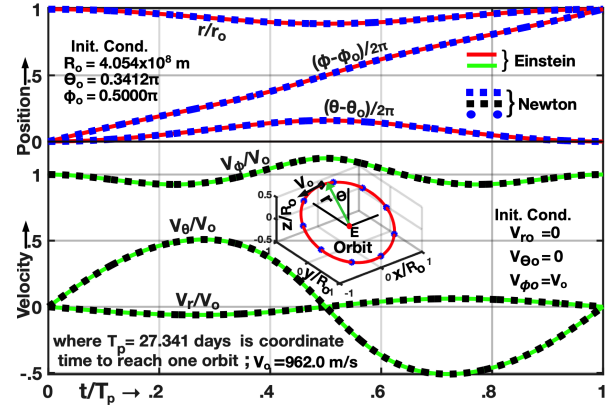


Figure 4 Moon Orbits Earth Traj.: Einstein vs Newton

taken at its apogee with $v_{\phi 0} = 962$ m/s: the initial polar $\theta_0 = 90^\circ - 23.44^\circ$ (earth obliquity) $- 5.14^\circ$ (lunar orbital incline) $= 61.42^\circ$ ($.3412\pi$) and the starting $\phi_0 = \pi/2$ & using Eq.(15), the startup time rate is $(dt/d\lambda)_0 = 1.000000000161$. The starting radius $R_0 = 4.05E8$ m is large compared to $R_s = 0.00886488$ m, thus the significance of the S term approaching ∞ never comes into play herein. Thus $S \approx 1/r$ which is why the GR and NULG solutions look so close. Both theories give essentially the same average integrated velocity $(\int |v| d\lambda)/T_p = 1019$ m/s; apogee/perigee ratio $r_a/r_p = 1.122$; and orbit period $T_p = 27.343$ days compared to Wikipedia's $\{1022$ m/s, 1.1159 , 27.332 days $\}$.

4.2 Ex.3 Einstein vs Newton Black Hole Plunge

A black hole with the mass of a KYstar ($M = 3.33E7 M_E$ compared to Earth's mass M_E , is collapsed into a concentrated mass where $R_s = 295,347$ m. The simulation starts near R_s (i.e. $r = R_0 = 1.5R_s$) with the same starting angles as Ex.2 (picked to make the solutions work harder to feel all r, θ, ϕ variables). The body is kept from falling mainly radial by applying a large tangential starting velocity $v_{\phi 0} = 0.4330c$ (see Fig.(9) inset).

The Exs.1-2 solutions operate at $r \gg R_s$ therefore the Fig.(1) approximate transition zones never come into play there. However in Exs. 3-4, we are well within the neighborhood where the transition function

$r = O(R_s)$ is felt. Without introducing the $S_n = T_n(r)$ transition zone, the numerical solution asymptotically tends towards $r = R_s$ (without actually breaking through) until it runs into numerical integration difficulty with increasing λ since $S = 1/(r - R_s) \rightarrow \infty$ as $r \rightarrow R_s$.

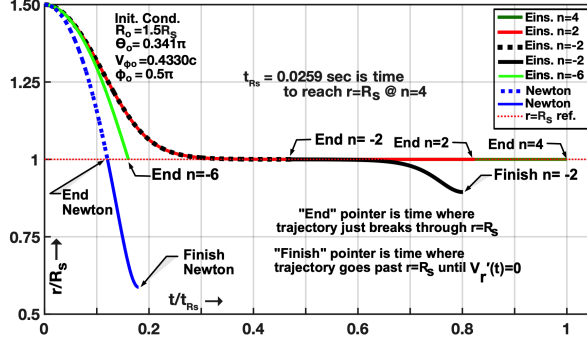


Figure 5 Radial r/R_s vs t/t_{R_s} Plunge to Black Hole

Figure(5) displays the normalized radial component r/R_s vs. normalized “coordinate time” t/t_{R_s} which uses the Fig.(1) transition functions $n=\{4,2,-2,-6\}$,

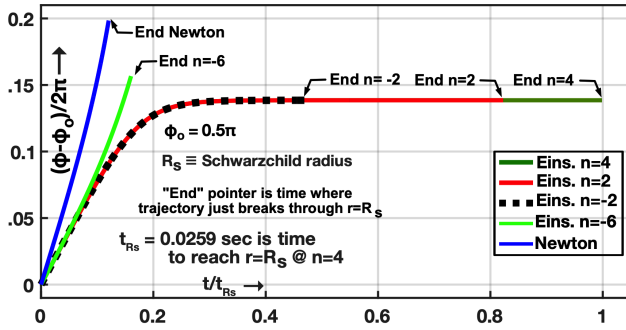


Figure 6 Azimuthal $\Delta\Phi/2\pi$ vs t/t_{R_s} Plunge to Black Hole

where GR is shown compared to Newton's solution. The vs. t plotting is stopped at the time increment *just before* crossing past $r = R_s$, except for $n = -2$, where using the full Fig.(1c) transition, the

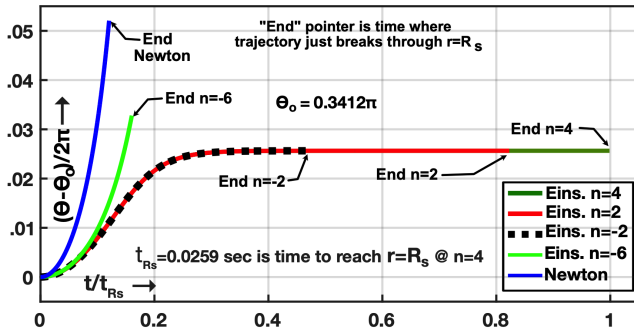


Figure 7 Polar $\Delta\theta/2\pi$ vs t/t_{R_s} Plunge to Black Hole

solution is also run inside the black hole for $r < R_s$. Plotting is stopped when it bottoms out at $v'_r(t) = 0$ (this happens at $r/R_s = 0.894$ where this point falls off of the Fig.(1c) plot domain, i.e. far to the left in the transition plot). For comparison, Newton's solution is similarly run out until it's $v'_r(t) = 0$.

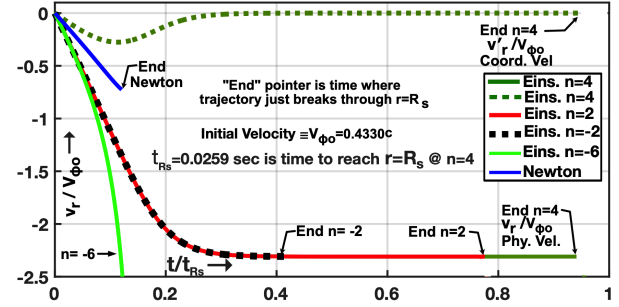


Figure 8a Radial $V_r/V_{\phi 0}$ vs t/t_{R_s} Plunge to Black Hole

The use of Schwarzschild coordinates, results in some odd behavior where the solution appears to flatten out in t as $r \rightarrow R_s$ (see for example Fig.(25.5) Ref.[6]) and is sometimes referred to as *coordinate freeze* Ref.[2]. In the words of Ref.[1], “This continues forever: we would never see the observer cross $r = R_s$, we would just see them move more and more slowly and become redder and redder as if embarrassed to have done something as stupid as diving into a black hole”.

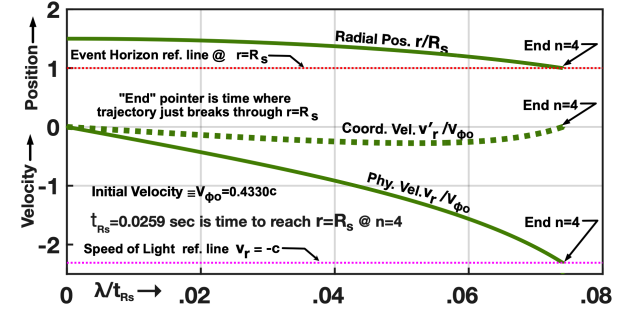


Figure 8b Radial $V_r/V_{\phi 0}$ vs λ/t_{R_s} Plunge to Black Hole

The same Fig.(5) r/R_s is replotted vs. “proper time” λ/t_{R_s} for $n=4$ in Fig.(8b), which gives a different look to the solution. The Fig.(10) nonlinear shape of the coord time t/t_{R_s} vs proper time λ/t_{R_s} curve as $t/t_{R_s} \rightarrow 1$ illustrates why the r/R_s flattens out where very small changes in λ/t_{R_s} corresponds to large changes in t/t_{R_s} . Normally, Schwarzschild coordinates are not intended to be used for $r < R_s$, however the solution inside the black hole $r < R_s$, can be solved using the Ref.[1]

$$ds^2 = g_{tt} dv^2 + 2dvdr + \sum_{\mu=\theta,\phi} g_{\mu\mu} d\mu^2 ; v \equiv ct + r + R_s \ln\left(\frac{r}{R_s} / g_{rr}\right) \quad (17)$$

Eq.(17) “*E-F metric*” (written in terms of the Eq.(10) Schwarzschild metric g_{kk} notation). Using this *E-F metric* is beyond the scope of this paper, and is only mentioned here to refer to what is sometimes used to get solutions inside the black hole.

Note that the bigger n (i.e. a stiffer S_n transition), the further out the solution goes in t , before crossing $r=R_s$. Also notice that for the very small $n=6$ (i.e. a very soft S_n transition), the position solutions (Figs.(5-7)) start to look more like the shape of Newton's solution. The $n=6$ transition is different from the other three in that its initial value r point is *inside* the transition, as indicated as red symbol $\circ$ on $n=6$ dotted curve in Fig.(2).

The shifted & normalized azimuthal and polar angular positions $(\phi-\phi_0)/2\pi$ & $(\theta-\theta_0)/2\pi$ are shown in Figs.(6-7), and exhibit the same flattened coordinate freeze shapes as the radial position.

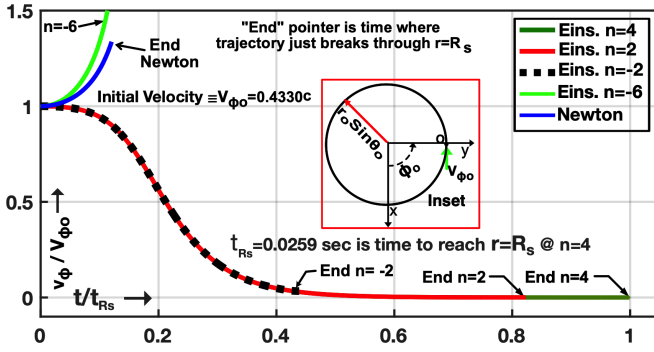


Figure 9 Azimuthal $V_\phi/V_{\phi 0}$ vs t/t_{R_s} Plunge to Black Hole

The $v_r/v_{\phi 0}$ (radial *physical velocity* normalized w.r.t. the initial velocity $v_{\phi 0}$) vs. t/t_{R_s} coordinate time is shown in Fig.(8a) & also vs. proper time λ/t_{R_s} in Fig.(8b). The radial *coordinate velocity* $v'_r/v_{\phi 0}$ is also shown in Fig.(8b)

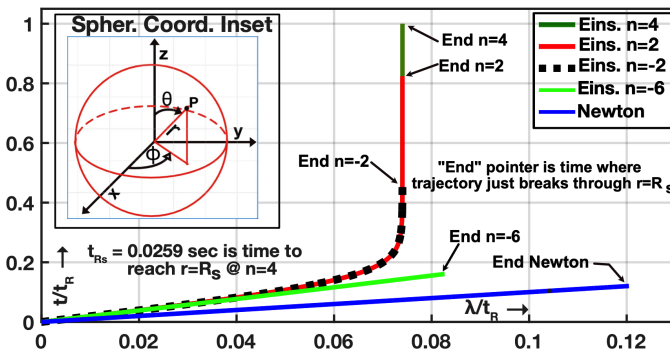


Figure 10 Coord. Time t/t_{R_s} vs λ/t_{R_s} Plunge to Black Hole

for comparison purposes, where it is noted that $v_r/v_{\phi 0}$ goes to a nonzero limit, whereas $v'_r/v_{\phi 0} \rightarrow 0$. The related azimuthal *physical velocity* $v_\phi/v_{\phi 0}$ vs. t/t_{R_s} is shown in Fig.(9). The

Eqs.(12b) *local physical velocity* $\vec{v}$ is measured by a static (hovering) observer at fixed $\{r, \theta, \phi\}$ whereas the *coordinate velocity* $\vec{v}'$ is measured by an observer at infinity; e.g. $r \rightarrow \infty$ in Eq.(13).

Expanding Eqs.(12b) reveals the relation between physical and coordinate velocity as $v_r = v'_r/(1-R_s/r)$ and $v_\phi = v'_\phi/\sqrt{(1-R_s/r)}$. Thus with $(1-R_s/r)$ appearing in the denominator, there is a *physical velocity* post-processing problem with dividing by zero when $r \rightarrow R_s$.

However because both the numerators v'_ϕ and v'_r also $\rightarrow 0$ as $r \rightarrow R_s$, we have a $0/0$ indeterminate form which has limits of 0 and c (e.g. $v_r/v_{\phi 0} = -2.31 \therefore v_r = -2.31 \times v_{\phi 0} = -2.31 \times 0.433c = -c$) and it is noted from Ref.[5], "the speed of a test particle, freely falling towards the event horizon of a black hole in Schwarzschild space-time tends to c ". The numerical integration does not experience this divide by zero, since with the aid of the transition S_n , it passes on through $r=R_s$, such as the Fig.(5) "Finish $n=-2$ " curve. The post-processing of raw output variables of $\{dr/d\lambda, d\phi/d\lambda, dt/d\lambda\}$ into $\{v_r, v_\phi\}$ via Eq.(12b) is where the sensitive limiting process takes place. We never compute at exactly $r=R_s$, but terminate the plots slightly before the $r=R_s$ point. Real

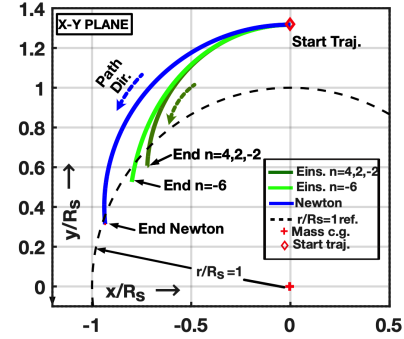


Figure 11 x-y Plane: Plunge Traj. to Black Hole

values of the physical velocity v_ϕ can't be experienced for $r < R_s$ because of the $\sqrt{}$ appearing in the above *in line text*

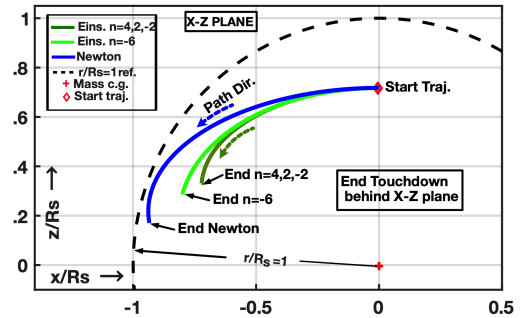


Figure 12 x-z Plane: Plunge Traj. to Black Hole

v_ϕ denominator formula which results in unrealistic complex values. This is a further reason why **E-F** coordinates are needed for computations inside the black hole.

The 3-D trajectories are shown in Figs.(11-13) and correspond to three projections on to the x-y, x-z, z-y planar cuts. The Figs.(12-13) paths appear to end inside $r=R_s$, however, that is an optical illusion because the surface intersection

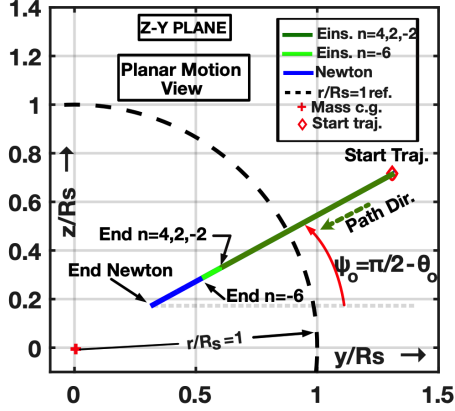


Figure 13 y-z Plane: Plunge Traj. to Black Hole

points are out of the planar cut, thus fall on the dashed curved surface behind the cut plane. The paths are traced out with λ as an advancement parameter, thus the three $n=\{4,2,-2\}$ trajectories all lie upon one another, except for a small imperceptible difference at the end. The Fig.(13) plot appears as a line since it sees the plane of motion as an edge at an angle of $\psi_0 = \pi/2 - \theta_0$. Say θ_0 was instead 0.5π , then the complete solution would be in the x-y plane and therefore would look as an edge when viewed from either the x-z or the y-z plane projections. However in our case $\theta_0=0.3412\pi$, therefore it is as though we rotated the xy plane solution about the x axis by $\psi_0=\pi/2 - \theta_0$ which is why we see the Fig.(13) rotated edge.

4.3 Ex. 4 Einstein vs Newton Black Hole Death Spiral

This example is exactly the same as the previous Ex. 3, except the starting initial position

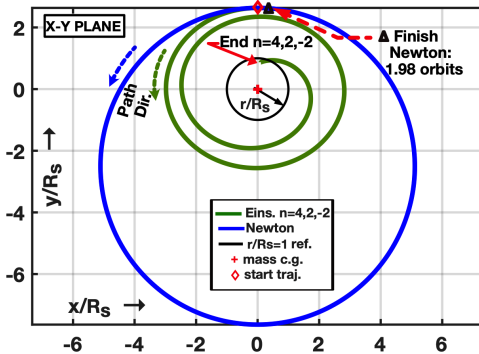


Figure 14 x-y Plane: Spiral Traj. to Black Hole

R_0 is twice as far away and the $v_{\phi 0}$ starting velocity is 15% bigger, which results in an orbiting death spiral situation. In the following narrative, most of the plot descriptive comments on the $r \rightarrow R_s$ approach features (e.g. frozen coordinates and indeterminate *physical velocities* etc.) would apply here as well and will not be repeated. The time history plots are better understood by viewing the Fig.(14-15) death spiral orbits first. The Fig.(14) x-y plane view shows a substantial difference between the GR orbit and the NUGL one. The GR death spiral solution travels for 1.98 orbits before (with the aid of the S_n transition function) crossing $r=R_s$, whereas the NUGL solution does not experience any such strong inward spiral where the integration is deliberately halted after it also runs for 1.98 orbits (note the NUGL orbits overlap thus no spiral). With the same initial conditions, it appears that the effect of gravity in GR is stronger than NUGL where GR pulls the falling mass inwards

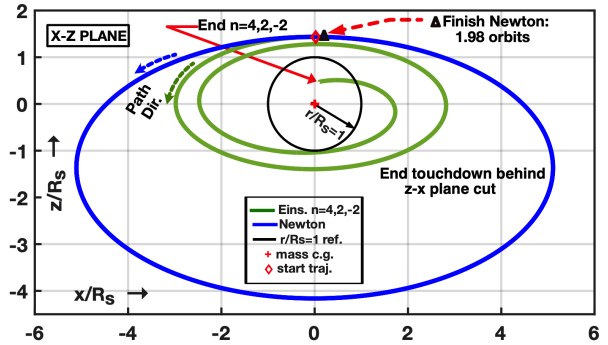


Figure 15 x-z Plane: Spiral Traj. to Black Hole

towards the black hole, whereas NUGL displays the trajectory as repeated overlapping orbits.

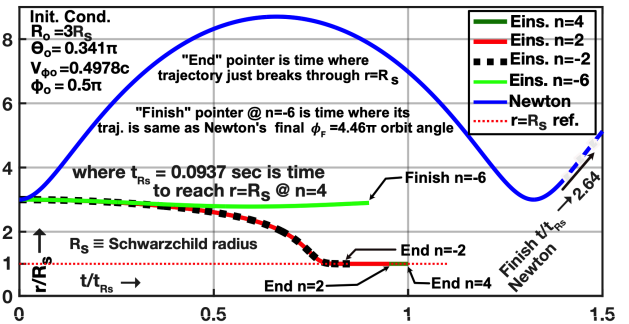


Figure 16 Radial r/R_s vs t/t_{R_s} Spiral to Black Hole

Figure 16 shows the radial trajectory vs t/t_{R_s} coordinate time, where using S_n transition functions $n=\{4,2,-2\}$ exhibit the GR death spiral however using the weakest $n=-6$,

exhibits properties closer to NUGL, where the mass would orbit rather than fall towards the black hole.

The related azimuthal & polar angular positions $(\phi - \phi_0)/2\pi$; $(\theta - \theta_0)/2\pi$ are shown in Figs.(17-18), and

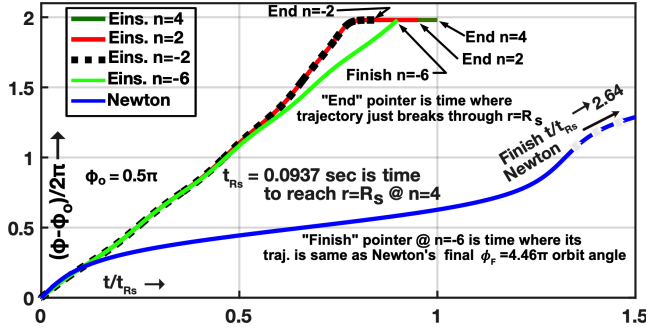


Figure 17 Azimuthal $\Delta\phi/2\pi$ vs t/t_{R_S} Spiral to Black Hole

exhibit the same flattened *coordinate freeze* shapes as the radial position. In Figs.(16-20), the NUGL solution is repetitive once the first orbit is completed, therefore we let Newton's second orbit fall off the page

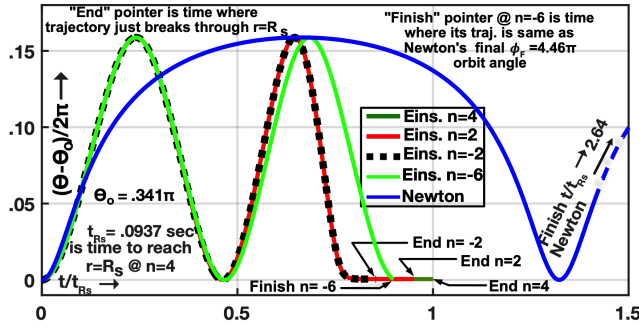


Figure 18 Polar $\Delta\theta/2\pi$ vs t/t_{R_S} Spiral to Black Hole

since it does not show any new useful information. The Fig.(18) $(\theta - \theta_0)/2\pi$ polar angular plots display the approximately 2 orbits for both the GR and the NUGL

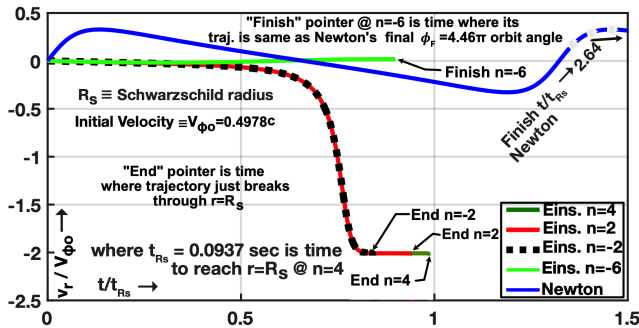


Figure 19 Radial V_r/V_{ϕ_0} vs t/t_{R_S} Spiral to Black Hole

solutions, however it is noted that due to Newton's oblong orbit, it takes much more time to complete the two cycles.

The v_r/v_{ϕ_0} (radial *physical velocity* normalized w.r.t. initial velocity v_{ϕ_0}) vs. coordinate t/t_{R_S} is shown in Fig.(19) and the corresponding azimuthal velocity v_{ϕ}/v_{ϕ_0} in Fig.(20).

Again as in the previous plunge Ex. 3, the $(1-R_S/r)$ and $\sqrt{(1-R_S/r)}$ terms appearing in the denominator of the v_r & v_{ϕ} , gives a *physical velocity* post-processing problem with dividing by zero when $r \approx R_S$

The last few time steps of the integration are omitted in the plots since the numerical solution sensitivity is not accurate enough to give the proper limit. The radial velocity terminates at $v_r = -c$ (e.g. $v_r/v_{\phi_0} = -2.009 \therefore v_r = -2.009 \times v_{\phi_0} = -2.31 \times 0.4978c = -c$) and the azimuthal velocity ends at $v_{\phi} = 0$.

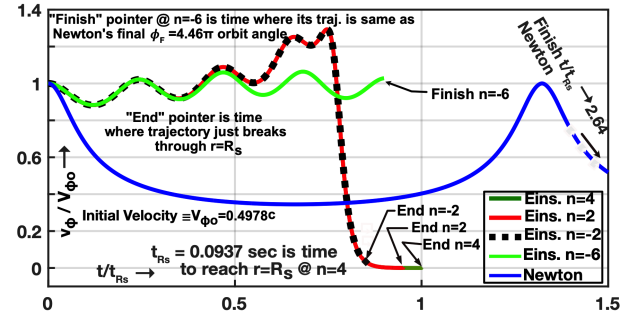


Figure 20 Azimuthal V_{ϕ}/V_{ϕ_0} vs t/t_{R_S} Spiral to Black Hole

5. Concluding Remarks

A sequence of 4 increasing complex examples were successfully solved for the GR and NUGL time histories. Free-fall trajectories in the Earth's gravitational field shows little difference between GR and NUGL, however for the geodesic paths in the neighborhood of strong gravitational fields like the KY-Star, the deviations are substantial.

6. References

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