

A Multiphysics Framework for Quantitative Assessment of Operational Resource Depletion

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Abstract

The existence of a large, or even apparently unlimited, external resource reservoir does not imply that a resource-dependent system can sustain an arbitrarily large rate of consumption. A distinction must be made between the amount of resource that exists and the rate at which that resource can be accessed, transported, and delivered to the region where it is required. This distinction becomes particularly relevant when demand increases with time. Even if the external resource reservoir remains permanently available, the mechanisms responsible for extracting or transporting the resource operate at finite rates. A consuming region may therefore become locally resource-limited while substantial resources remain available elsewhere. The present work develops a spatially resolved framework for investigating this mechanism. The model is deliberately generic: the transported quantity may represent matter, an energy carrier, nutrients, raw materials, or any other resource whose use requires transfer from an external reservoir toward a localized consumer.

Keywords: resources accessibility, finite transport capacity, technological efficiency.

Introduction

The system combines three physical processes: internal redistribution of the resource, finite-rate access from an external reservoir, and localized consumption driven by a time-dependent demand. Technological efficiency is subsequently incorporated to determine whether improved resource utilization can eliminate the accessibility constraint or only postpone its onset. Finally, uncertainty is introduced into the external access rate in order to study how fluctuations in supply capacity propagate into the operational lifetime of the system. The central hypothesis is that operational depletion can arise without physical exhaustion of the external resource. If demand grows while the maximum rate of resource delivery remains finite, the consuming region can eventually become transport-limited. A bounded technological efficiency may delay this transition, but it cannot indefinitely compensate for increasing resource requirements. When access is stochastic, the corresponding depletion horizon becomes a random variable rather than a unique deterministic time. The theoretical model is first formulated analytically. It is subsequently implemented as a time-dependent spatial problem in COMSOL Multiphysics®, with LiveLink™ for MATLAB®

used for parameter exploration, stochastic forcing, automated model execution, and statistical post-processing.

Theory / Experimental Set Up

Governing transport equation

Let $C(x, t)$ denote the local resource availability or concentration. Conservation of the resource inside the computational domain is described by

$$\frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C) - Q_{\text{cons}}(x, t) \quad (1)$$

where D is the effective transport coefficient and Q_{cons} is the local consumption rate.

The diffusion term redistributes the resource through the domain, whereas the sink term removes resource in the consumption region. The evolution is therefore controlled by the competition between spatial replenishment and local extraction.

Axisymmetric formulation

The numerical problem is formulated in a two-dimensional axisymmetric domain using coordinates (r, z) . Under axisymmetry, Eq. (1) becomes

$$\frac{\partial C}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r D \frac{\partial C}{\partial r} \right) + \frac{\partial}{\partial z} \left(D \frac{\partial C}{\partial z} \right) - Q_{\text{cons}} \quad (2)$$

The three-dimensional volume associated with the meridional computational domain Ω is recovered through the axisymmetric measure

$$\int_V f dV = 2\pi \int_{\Omega} r f(r, z) d\Omega. \quad (3)$$

Equation (3) will subsequently be used to define the global resource inventory and total resource consumption. It will not be reintroduced in the numerical or results sections.

Localized resource consumption

Consumption is confined to a localized region through a dimensionless spatial weighting function $\chi(x)$, which approaches unity in the consuming region and vanishes away from it.

Resource extraction must also decrease when the local resource availability becomes small. This effect is represented through the saturating factor

$$f(C) = \frac{C}{K_m + C}, \quad (4)$$

where K_m is a characteristic resource concentration. The local consumption law is therefore defined as

$$Q_{\text{cons}}(\mathbf{x}, t) = \chi(\mathbf{x}) q_{\text{eff}}(t) \frac{C}{K_m + C}. \quad (5)$$

Here $q_{\text{eff}}(t)$ denotes the resource requirement after technological efficiency has been taken into account. Its explicit definition is introduced in the following section and will thereafter be used throughout the manuscript without redefining Eq. (5), that distinguishes the requested resource requirement from the resource consumption that can actually be realized. In the resource-rich limit, $C \gg K_m$ the saturation factor approaches unity and $Q_{\text{cons}} \approx \chi q_{\text{eff}}$. In contrast, when $C \ll K_m$, the local consumption rate decreases approximately in proportion to C . Hence, $Q_{\text{cons}} \rightarrow 0$ as $C \rightarrow 0$. This prevents the model from extracting finite quantities of resource from an already depleted region and creates a direct coupling between resource availability and realizable throughput.

Growing useful demand

The externally imposed useful demand is represented by the exponential law

$$q_{\text{dem}}(t) = q_0 e^{gt}, \quad (6)$$

where q_0 is the initial demand and g is its growth rate. The exponential form is used as a controlled forcing function rather than as an assumption that every real system can grow exponentially indefinitely. Its purpose is to generate a progressively increasing resource requirement and thereby expose the transition between a regime in which transport can satisfy demand and one in which resource accessibility becomes the dominant constraint.

The characteristic demand-growth time is $\tau_g = 1/g$. This time scale will later be compared with the characteristic transport, access, technological, and stochastic correlation times. At this stage, Eq. (6) specifies the useful demand only. The amount of physical resource required to sustain that demand depends on technological efficiency and is introduced next.

Finite Resource Access, Technological Efficiency, and Operational Depletion

The governing transport model introduced earlier in Eq. (1) describes the redistribution and localized consumption of the resource. We now introduce the two mechanisms that determine whether the consuming region can remain operational under growing demand: finite access to the external reservoir and technological efficiency.

Finite-rate access to an external resource reservoir

The external environment is represented by a reservoir maintained at a constant resource concentration C_{ext} . The reservoir is assumed to remain available throughout the simulation; therefore, the model does not impose physical exhaustion of the external resource.

Access to this reservoir is nevertheless finite.

Resource transfer across the external boundary is described by the Robin condition $J_{\text{in}} = h_{\text{acc}} (C_{\text{ext}} - C)$, where J_{in} is the inward resource flux and h_{acc} is an effective access or transfer coefficient.

This equation introduces the fundamental distinction between *resource abundance* and *resource accessibility*. The external resource may remain permanently available, while the rate at which it can enter the system remains finite.

Since the resource concentration is non-negative $C \geq 0$, the above relation immediately implies the upper bound $J_{\text{in}} \leq J_{\text{in,max}} = h_{\text{acc}} C_{\text{ext}}$. For an accessible boundary of finite area A_b , the corresponding total supply capacity satisfies $M_{\text{in}} \leq M_{\text{in,max}} = A_b h_{\text{acc}} C_{\text{ext}}$. Thus, even an effectively unlimited external reservoir supplies the system through a finite transport capacity. This rate limitation, rather than the total amount of resource

outside the domain, constitutes the central constraint considered in the present model.

Bounded technological efficiency

Technological improvement is represented by a dimensionless efficiency $\eta(t)$. For the useful demand defined by Eq. (6), the corresponding physical resource requirement is

$$q_{\text{eff}}(t) = \frac{q_{\text{dem}}(t)}{\eta(t)}. \quad (7)$$

Increasing efficiency therefore reduces the amount of physical resource required to provide the same useful service.

An unbounded efficiency law would allow technological progress to compensate mathematically for arbitrarily large demand. Such an assumption would embed unlimited technological improvement directly into the model and would therefore obscure the physical accessibility constraint that we seek to investigate. We instead adopt the bounded law

$$\eta(t) = \eta_0 + (\eta_{\text{max}} - \eta_0)(1 - e^{-\alpha t}), \quad (8)$$

where η_0 is the initial efficiency, η_{max} is the maximum attainable efficiency, and α determines the rate of technological improvement. This definition satisfies

$$\eta(0) = \eta_0, \quad \lim_{t \rightarrow \infty} \eta(t) = \eta_{\text{max}}.$$

The associated technological time scale is

$$\tau_\eta = \frac{1}{\alpha}.$$

Combining Eqs. (6), (7) and (8) gives the explicit effective resource requirement

$$q_{\text{eff}}(t) = \frac{q_0 e^{gt}}{\eta_0 + (\eta_{\text{max}} - \eta_0)(1 - e^{-\alpha t})}. \quad (9)$$

Accordingly, the consumption term appearing in the governing PDE, Eq. (1), is completely specified by substituting Eq. (9) into the consumption law Eq. (5).

Use of Simulation

The stochastic trajectories are generated in MATLAB before each COMSOL simulation. For a temporal discretization $t_n = n\Delta t$, the Ornstein–Uhlenbeck process can be advanced using its exact discrete transition,

$$X_{n+1} = e^{-\Delta t/\tau_h} X_n + \sigma_h \sqrt{1 - e^{-2\Delta t/\tau_h}} \xi_n$$

Where $\xi_n \sim N(0,1)$ are independent standard Gaussian random variables.

The numerical strategy combines COMSOL Multiphysics® with MATLAB® through LiveLink™ for MATLAB®. This division is particularly suitable for the present problem because the two platforms perform complementary tasks. COMSOL is used to solve the spatially distributed time-dependent transport problem, whereas MATLAB® controls the parameter studies, generates the stochastic processes, executes repeated simulations, extracts the relevant physical observables, and performs the subsequent statistical analysis.

The central objective of the simulations is to determine whether a system connected to a permanently available external resource reservoir can nevertheless undergo operational depletion when the rate at which the resource can enter and propagate through the system is finite. The external resource is therefore not assumed to disappear during the simulation. Instead, the limitation arises from its finite accessibility. Within this framework, technological progress modifies the amount of resource required to sustain a prescribed useful demand, while the physical transport mechanism determines whether this requirement can actually be supplied to the consuming region.

Global resource balance

To distinguish local operational depletion from complete resource exhaustion, we introduce the total resource inventory contained within the

$$M(t) = 2\pi \int_{\Omega} r C(r, z, t) d\Omega.$$

The total instantaneous consumption rate is

$$\dot{M}_{\text{cons}}(t) = 2\pi \int_{\Omega} r Q_{\text{cons}}(r, z, t) d\Omega.$$

Integrating the conservation equation over the complete physical volume gives the global balance

$$\frac{dM}{dt} = \dot{M}_{\text{in}}(t) - \dot{M}_{\text{cons}}(t). \quad (10)$$

that provides an important physical interpretation of the transient dynamics. Whenever $\dot{M}_{\text{cons}} > \dot{M}_{\text{in}}$, the deficit is supplied by the resource already stored

inside the domain, and therefore $\frac{dM}{dt} < 0$.

The internal inventory consequently acts as a temporary buffer between growing demand and finite external access.

Integrating Eq. (10) from 0 to t gives

$$M(t) = M(0) + \int_0^t [\dot{M}_{in}(s) - \dot{M}_{cons}(s)] ds.$$

Operational depletion is therefore history-dependent. A temporary supply-consumption imbalance need not immediately produce failure if the domain contains sufficient stored resource. Persistent deficits, however, progressively consume this buffer.

Operational depletion criterion

The relevant failure mechanism in the present model is not defined by $M(t)=0$. Instead, the system becomes operationally depleted when resource availability in the consuming region falls below a prescribed level. Let $C_c(t) = C(r=0, z=0, t)$ denote the concentration at the center of the consumption region. For a critical availability C_{crit} , we define

$$T_{dep} = \inf\{t > 0 : C_c(t) \leq C_{crit}\}. \quad (11)$$

For the simulations considered in this work, $C_{crit}=0.1$. This definition makes it possible for

$$C_c(T_{dep}) \leq C_{crit} \quad \text{while} \quad M(T_{dep}) > 0.$$

Thus,

$$\text{operational depletion} \neq \text{global resource exhaustion}.$$

A finite quantity of resource may remain inside the domain, and the external reservoir may remain permanently available, while the consumer is unable to obtain that resource at the location and rate required for continued operation.

Deterministic prediction

The preceding analysis leads to a clear prediction for the deterministic simulations. Increasing the technological-improvement rate α reduces the resource requirement during the transient toward η_{max} . This reduces cumulative consumption, preserves the internal inventory, and therefore increases the operational depletion time. Hence, within the parameter regime considered,

$$\alpha \uparrow \Rightarrow T_{dep} \uparrow.$$

However, once $\tau_\eta \ll T_{dep}$, efficiency reaches η_m sufficiently early that further increases in α no longer significantly modify the subsequent resource dynamics. The effective requirement then

approaches $q_{eff}(t) \simeq \frac{q_0}{\eta_{max}} e^{gt}$, and the operational lifetime becomes controlled predominantly by demand growth and finite transport accessibility. We therefore expect

$$\lim_{\alpha \rightarrow \infty} T_{dep}(\alpha) = T_{dep}^{(\infty)} < \infty.$$

This prediction is tested directly by the deterministic COMSOL simulations presented later.

COMSOL Multiphysics® model

The complete deterministic COMSOL model can therefore be summarized by

$$\begin{aligned} \frac{\partial C}{\partial t} &= \nabla \cdot (D \nabla C) - \chi(r, z) \frac{q_0 e^{gt}}{\eta(t)} \frac{C}{K_m + C}, \\ \eta(t) &= \eta_0 + (\eta_{max} - \eta_0)(1 - e^{-\alpha t}), \\ J_{in} &= h_{acc}(C_{ext} - C), \\ C(r, z, 0) &= C_{init}. \end{aligned}$$

The formulation deliberately retains the complete spatial PDE rather than reducing the system to a single ordinary differential equation.

Extraction of global quantities

In addition to the central concentration, MATLAB® extracts spatially integrated quantities from the COMSOL solution. Total accessible resource inventory is evaluated as

$$M(t) = 2\pi \int_{\Omega} r C(r, z, t) d\Omega,$$

where the factor $2\pi r$ reconstructs the physical three-dimensional volume from the two-dimensional axisymmetric computational domain. Similarly, the total instantaneous resource consumption is calculated as

$$\dot{M}_{cons}(t) = 2\pi \int_{\Omega} r Q_{cons}(r, z, t) d\Omega.$$

The numerical workflow therefore produces three complementary classes of information, $C_c(t)$, $M(t)$, $\dot{M}_{cons}(t)$, representing, respectively, local resource availability, global accessible inventory, and actual resource throughput. This distinction is important because the occurrence of $C_c(t) \leq C_{crit}$ does not necessarily imply $M(t) \approx 0$.

MATLAB[®] post-processing therefore allows local operational depletion to be compared directly with the global resource state.

The COMSOL–MATLAB[®] coupling is therefore not used merely as a convenient post-processing interface. It constitutes an essential part of the numerical methodology. COMSOL retains the spatially resolved conservation problem, whereas MATLAB[®] converts that model into a controlled numerical experiment in which technological improvement and stochastic accessibility can be systematically explored.

Simulation Results

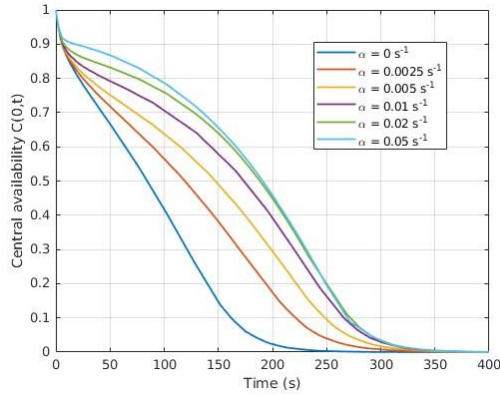


Figure 1. Temporal evolution of the central resource availability $C_c(t)$ for different technological-improvement rates α .

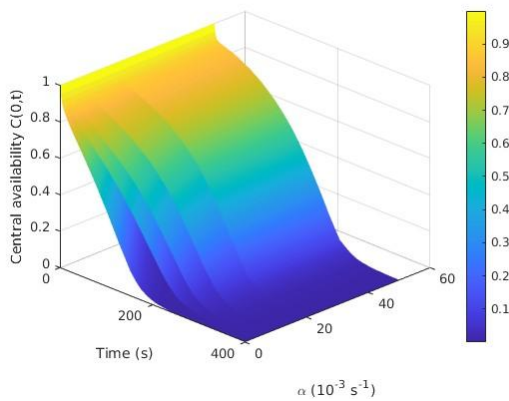


Figure 2. Three-dimensional representation of the central resource availability as a function of time and technological improvement rate, $C_c = C_c(t, \alpha)$.

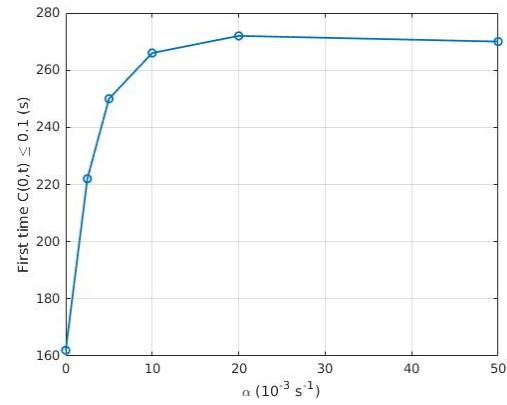


Figure 3. Operational depletion time T_{dep} as a function of the technological-improvement rate α .

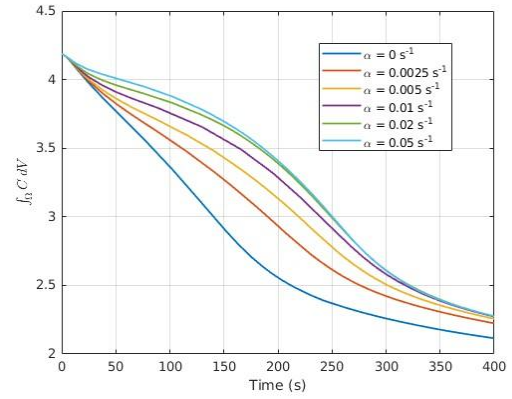


Figure 4. Evolution of the total amount of the simulated resource available inside the considered domain.

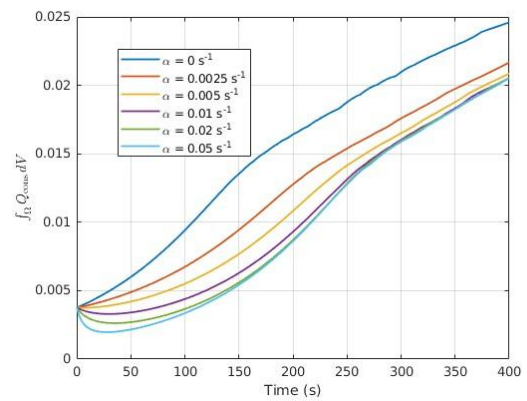


Figure 5. Evolution of the total consumption in the domain of the simulation.

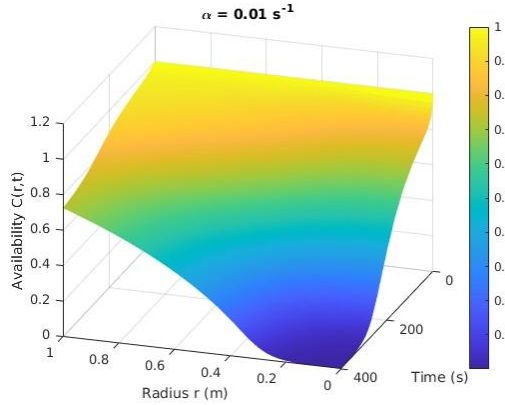


Figure 6. Radial resource-availability profiles at selected times during the deterministic simulation. The initially nearly uniform resource field progressively develops a depleted central region as localized consumption increases.

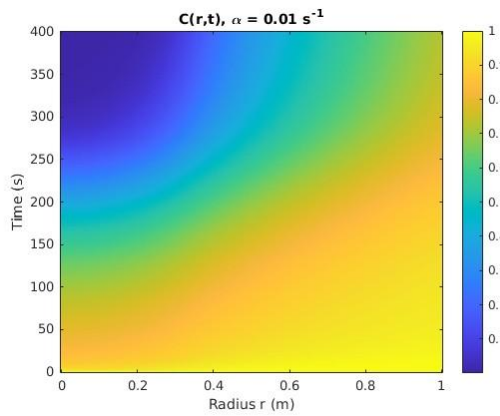


Figure 7. Density map of the spatiotemporal resource field $C(r, t)$. Resource depletion begins near the localized central consumer and subsequently expands toward larger radii, while the outer part of the domain remains comparatively resource-rich. Iso-concentration contours can therefore be interpreted as moving depletion fronts.

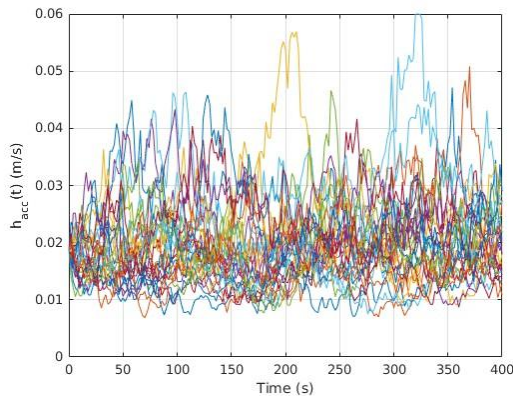


Figure 8. Ensemble of stochastic resource-access coefficients $h_{acc}(t, \omega)$ generated in MATLAB[®] using the log-transformed Ornstein-Uhlenbeck process.

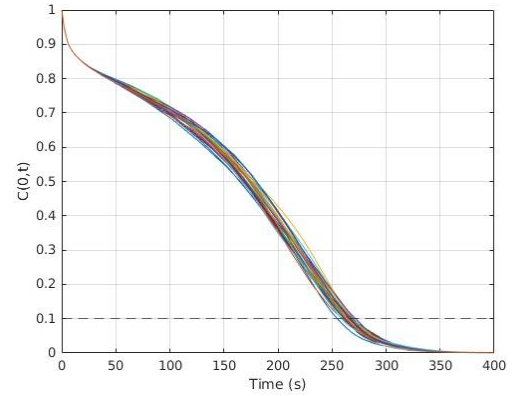


Figure 9. Ensemble of central resource-availability trajectories $C_c(t; \omega)$ obtained by propagating the stochastic resource-access histories through the complete spatial transport model.

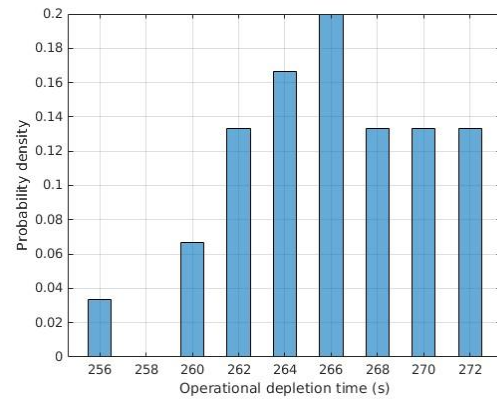


Figure 10. Empirical distribution of operational depletion times obtained from the stochastic COMSOL-MATLAB[®] ensemble. Each sample corresponds to the first time at which the central resource availability crosses the prescribed operational threshold. Fluctuations in external accessibility therefore transform the unique deterministic depletion horizon into a distribution of possible operational lifetimes.

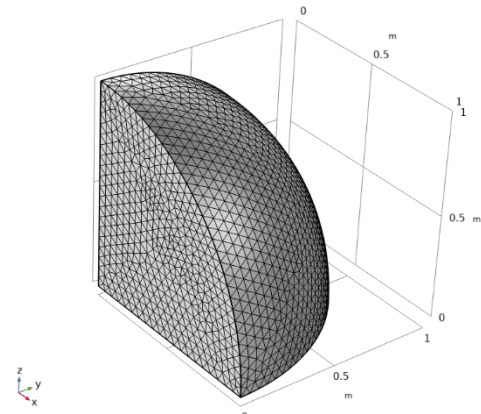


Figure 11. 3D-mesh of an sphere's octant where all the simulations could take place, nevertheless a 2D computation is also completely recommendable, obtaining the same results in case of assuming axial-symmetry.

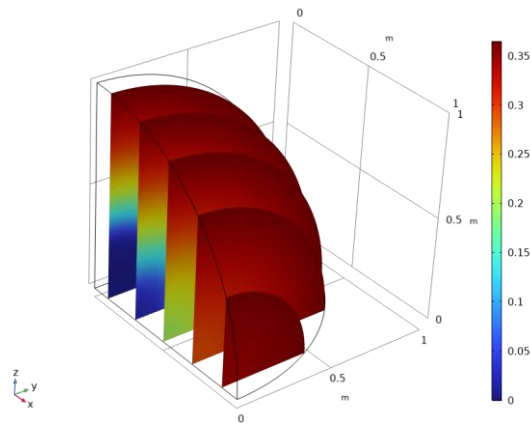


Figure 12. Parallel 2D cross sections of the 3D simulation carried out

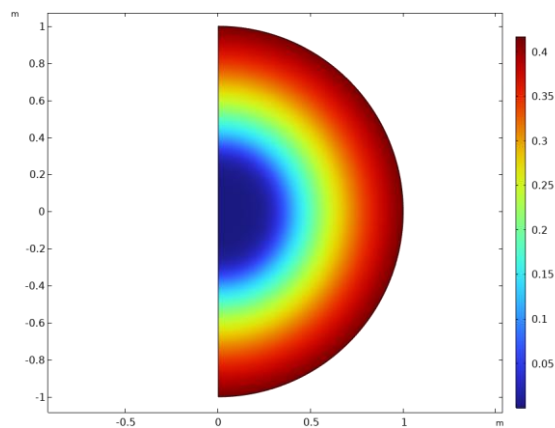


Figure 13. One of the 2D simulations carried out, this time with axial symmetry in a semi-circle.

Conclusions

Figure 1 shows the temporal evolution of the resource concentration at the center of the consumption region for the different technological-improvement rates considered in the deterministic parameter sweep.

The quantity represented is the central concentration $C_c(t)$, defined previously just before Eq. (11). Therefore, no additional state variable is introduced here: the figure directly extracts the local resource availability experienced by the consumer from the complete spatial solution $C(r,z,t)$.

All simulations start from the same initial condition and use identical values of the transport coefficient, external resource concentration, access coefficient, demand-growth rate, and maximum technological efficiency. The only parameter modified among the curves is the technological-improvement rate α . The results reveal three clearly differentiated stages of evolution. At early times, all trajectories remain close to one another and the central resource

concentration changes only moderately. In this regime, the resource requirement is still sufficiently small for external replenishment and redistribution of the internal inventory to compensate for consumption. The system is therefore operating far from its accessibility limit.

As demand increases, the trajectories begin to separate. The case $\alpha=0$, corresponding to the absence of technological improvement, experiences the earliest reduction in central resource availability. For $\alpha>0$, efficiency progressively approaches its upper bound $\eta_{\max}$, reducing the physical resource requirement associated with the same useful demand. Consequently, increasing α shifts the decline of $C_c(t)$ toward later times.

This displacement provides the first direct numerical confirmation that technological improvement can delay operational depletion. Faster improvement reduces cumulative resource extraction during the technological transient, preserving both the internal inventory and the concentration in the consuming region.

The effect, however, is not proportional to α .

In any case:

externally abundant resource
+ finite access and transport capacity
+ growing resource requirement
+ bounded technological efficiency
=> finite operational horizon;

Whereas stochastic accessibility
=> probabilistic operational horizon

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